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THREE-DIMENSIONAL STRESS CONCENTRATION AROUND A CYLINDRICAL HOLE IN A SEMI-INFINITE ELASTIC BODY

by

Carl K. Youngdahl and Eli Sternberg

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by

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ABSTRACT

We present a three-dimensional solution, exact within classical elastostatics, for the stresses and deformations arising in a half-space with a semi-infinite transverse cylindrical hole; if the body--at infinite distances from its cylindrical boundary--is subjected to an arbitrary, uniform plane field of stress that is parallel to the bounding plane. The solution in integral form was deduced with the aid of the Papkovich stress functions by means of an especially adapted, unconventional, integral-transform technique. Numerical results for the nonvanishing stresses along the boundary of the hole and for the normal displacement at the plane boundary, corresponding to several values of Poisson's ratio, are also included. These results exhibit in detail the three-dimensional stress boundary layer that emerges near the edges of the hole in the analogous problem for a plate of finite thickness, as the ratio of the plate thickness to the diameter of the hole grows beyond bounds. The results obtained illustrate the limitations inherent in the two-dimensional plane-strain treatment of the spatial plane problem; in addition, they are relevant to failure considerations and are of interest in connection with experimental stress analysis.

I. THEORETICAL ANALYSIS

A. Introduction

The exact solution of three-dimensional elasticity problems is usually difficult, particularly when stress concentrations due to geometric, material, or load discontinuities are involved. Unfortunately, approximate formulations or methods of solution blur or omit completely the effect of the stress concentration.

The half-space problem solved here falls into the general category of plane problems in elasticity. An extensive literature exists concerning the formulation of two-dimensional approximate theories and the mathematical techniques devised to obtain solutions to the associated problems.* However, without exact solutions for comparison, there is no way of determining the accuracy, and thereby the limitations, of the approximate two-dimensional theories and solutions. With reference to the particular problem discussed here, the approximate plane solutions give meaningful results for the stress states at points remote from the edge created by the intersection of the hole and the surface of the half-space, but their accuracy decreases as the edge is approached. For example, the three-dimensional solution obtained in this report predicts a large shearing stress just beneath the plane surface; this is completely missed by the associated two-dimensional approximate solution.

Various schemes for treating the three-dimensional aspects of the plane problem are given in Refs. 3-5. The particular problem of the stress concentration around a transverse circular hole in an infinite plate under uniform loads at infinity is discussed in Refs. 6-9. These treatments are based on the plate thickness to hole diameter ratio being a relatively small number; as the ratio approaches infinity, the stress-concentration effect becomes progressively more pronounced and localized near the surfaces of the plate, necessitating a separate approach and leading to the formulation of the problem discussed in this report. The solution for the axially symmetric problem of the half-space with a transverse cylindrical hole loaded by a uniform band of pressure applied at the entrance of the hole is presented in Refs. 10 and 11; the method of solution of the problem is similar to that employed here, the main differences being due to the problem discussed in this report not being axially symmetric.

It may be well to mention some of the stumbling blocks that make the solution of the problem difficult. 1) Since an exact three-dimensional solution is desired, the specialized mathematical approaches involving the use of conformal mapping, complex variables, or potential functions developed for the solution of problems in the plane approximate theories cannot be used. 2) The problem is not axially symmetric; hence the simplifications inherent in this symmetry and the techniques developed to take advantage of them cannot be made use of here. 3) Standard transformation methods are not applicable.

An arbitrary uniform plane state of stress can be expressed as the sum of a plane hydrostatic state of stress and a plane state of pure shearing stress. The elasticity problems for the half-space with a hole acted upon by each of these component plane stress states at infinity can be solved separately. Since we are working within the framework of linear elasticity, the

---

*See Refs. 1 and 2 for extensive bibliographies on three-dimensional stress concentrations and the plane problem, respectively.

results, properly weighted, can then be superimposed to give the solutions corresponding to an arbitrary uniform plane stress state, parallel to the bounding plane, at points remote from the hole. The solution for the plane hydrostatic state of stress at infinity can be obtained quite easily; in fact, the plane stress solution is, in this case, not an approximation, but the exact solution. Consequently, the solution of the problem for the plane state of pure shear at infinity is the principal objective of this work. It is conveniently formulated as the sum of the known plane-strain solution and the solution to the residual problem. The plane-strain solution, which violates the boundary conditions on the plane surface of the half-space but satisfies the boundary conditions on the hole and the governing differential equations, is a good approximation for the stress state at points remote from the bounding plane. Consequently, the solution to the residual problem gives the deviation of the plane-strain solution from the exact solution as the surface of the half-space is approached. The solution of the pure shear problem thus supplies two important types of information: 1) it shows what the exact stress distribution is near the surface of the half-space, and in particular, the distribution near the edge where the hole intersects the surface; and 2) it shows how far it is necessary to be from this surface for the plane-strain solution to be a useful approximation.

Numerical results are presented for the exact three-dimensional solution of the pure shear problem. In addition, these are added to the simple solution to the plane hydrostatic problem in such a way as to give the exact solution for the problem of uniform uniaxial tension at infinity, since this is probably the loading of most engineering interest. In other words, the stress-concentration effect of a hole in a thick plate being pulled in one direction is determined. The stress states at and near the edge are found to be strongly dependent on the Poisson's ratio of the elastic material, which is of particular significance for photoelastic work, where experiments must usually be performed with materials of Poisson's ratios different than those of common structural materials.

In Part I of this report, the elasticity problem is formulated for the half-space with a transverse hole, acted on, at points remote from the hole, by an arbitrary uniform plane field of stress parallel to the plane surface. This problem is broken up into component problems, corresponding to a plane hydrostatic state and a plane state of pure shear at infinity. The elementary solutions for the plane hydrostatic problem and the plane-strain solution associated with the pure shear problem are then presented.

The theoretical solution of the residual problem of pure shear, which comprises the bulk of the effort reported here, is developed next. The residual problem is formulated in terms of the Papkovich stress functions, specialized to provide the proper angular variation for the pure shear state. It is interesting to note that three stress functions are needed to provide a complete solution for this problem, as contrasted to axially symmetric problems for which it is known that only two are required. Pseudo-transform techniques are then employed.* In essence, general solutions are constructed to the partial differential equations for the stress functions which are not obtainable by transform methods or separation of variables. These solutions are of such a form that, upon substitution into the boundary conditions, they yield relations which have the same structure as the transformed boundary conditions obtained from the application of transformation methods in other fields. From this point on, the usual transform techniques can be employed. Fourier sine and cosine transforms are taken with respect to the coordinate measured along the axis of the hole. A transformation based on a relation similar to that proved in Weber's Integral Theorem** is derived; one integral in the corresponding pair of transform relations has limits of one and infinity, and is therefore suitable for the transformation in the radial direction. The application of these pseudo-transform methods results in a single integral equation, which is solved numerically with a digital computer. The final expressions for the stresses and displacements are integrals whose integrands involve known functions and the function which is the solution of the integral equation.

The numerical solution of the integral equation, the numerical check of the boundary conditions, and the calculation of the desired stress distributions is reported in Part II. Unfortunately, the calculation of the required integrals is not amenable to straightforward numerical integration methods. This necessitates the reformulation of the integral representations. In general, functions are determined which can be integrated in closed form and have the same asymptotic behavior as the integrands of the desired integrals. The differences between the corresponding pairs of integrands and functions are then found to be numerically integrable within appropriate limits of accuracy. The determination of these functions necessitates extensive investigation of the integrands of the original integral representation and probably as much or more effort as the entire theoretical solution of the problem.

Appendix A summarizes properties of the Bessel functions and related functions which were used in the solution of the problem. In Appendix B are derived numerous asymptotic expansions needed in the numerical analysis of the solution. The proof of the modified form of Weber's Integral Theorem is discussed in Appendix C. Appendix D contains the computer programs which were written to accomplish the desired numerical calculations.

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*The boundary conditions of the problem expressed in terms of the stress functions cannot be transformed, so ordinary transform methods cannot be used.

**See Ref. 13, p. 468, and Appendix C.

B. Statement of the General Problem

Consider the Cartesian system of coordinates x_1, x_2, and x_3. Let the half-space $x_3 \geq 0$ be occupied by a homogeneous, isotropic, elastic material having shear modulus μ and Poisson's ratio ν. This body is in a state of uniform plane stress parallel to the boundary $x_3 = 0$ and oriented such that x_1 and x_2 are principal axes. In other words, if we denote the components of the corresponding Cartesian stress tensor by τ_{ij}, i,j = 1, 2, 3, then

$$\tau_{11} = \tau_1; \quad \tau_{22} = \tau_2; \quad \tau_{12} = 0;$$
$$\tau_{33} = \tau_{32} = \tau_{31} = 0 \quad \text{for } x_3 \geq 0, \tag{1}$$

where τ_1 and τ_2 are constants.

Consider now the disturbance of this uniform stress field caused by the presence of a circular hole of radius a whose axis coincides with the x_3 axis (see Fig. 1). The stress state given by Eqs. (1) will hold as r goes to infinity, but we must now also impose the boundary conditions

$$\tau_{rr} = \tau_{r\theta} = \tau_{rz} = 0$$
$$\text{on } r = a,\ z \geq 0; \tag{2}$$

$$\tau_{zz} = \tau_{\theta z} = \tau_{rz} = 0$$
$$\text{on } z = 0,\ r \geq a, \tag{3}$$

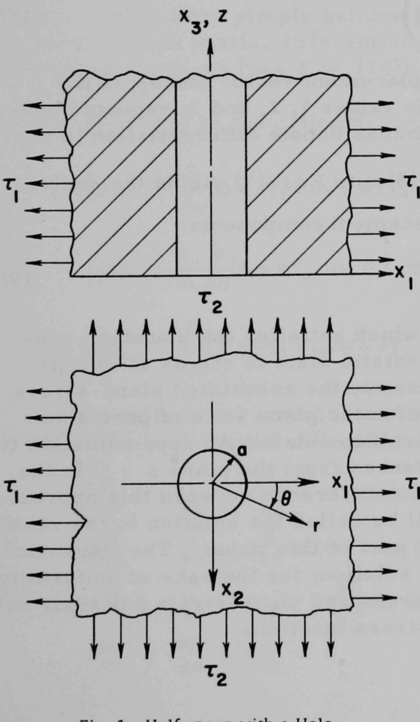

Fig. 1. Half-space with a Hole, in a State of Plane Stress

where τ_{rr}, $\tau_{r\theta}$, etc. are the components of the stress tensor in cylindrical coordinates r, θ, and z. The solution of the problem is conveniently expressed as the sum of the solutions for two basic loadings at infinity, namely, a plane hydrostatic state of stress and a plane state of pure shearing stress. The plane hydrostatic state of stress at infinity is characterized by

$$\left.\begin{array}{l}\tau_{11} = \tau_{22} = \tau_H; \quad \tau_{12} = 0; \\ \tau_{33} = \tau_{32} = \tau_{31} = 0\end{array}\right\} \text{at } r = \infty,\ z \geq 0; \tag{4}$$

the plane state of pure shear at infinity is characterized by

$$\left.\begin{aligned} &\tau_{11} = -\tau_{22} = \tau_S; \quad \tau_{12} = 0; \\ &\tau_{33} = \tau_{32} = \tau_{31} = 0 \end{aligned}\right\} \text{ at } r = \infty,\ z \geq 0. \tag{5}$$

The solution to the general problem corresponding to the boundary conditions (1), (2), and (3) is then equal to the sum of the solutions to the problems corresponding to the sets of boundary conditions (4), (2), (3) and (5), (2), (3) if

$$\tau_H = \tfrac{1}{2}(\tau_1 + \tau_2); \qquad \tau_S = \tfrac{1}{2}(\tau_1 - \tau_2). \tag{6}$$

The field equations which must be satisfied by a solution to an elasticity problem are the equations of equilibrium (for zero body forces)

$$\tau_{ij,j} = 0 \tag{7}$$

and the stress-displacement relations

$$\tau_{ij} = \mu\left(\frac{2\nu}{1-2\nu}\,\delta_{ij}u_{k,k} + u_{i,j} + u_{j,i}\right), \tag{8}$$

where u_i are the components of the displacement vector and δ_{ij} is the Kronecker delta. Subscripts take on the values 1, 2, and 3; repeated subscripts are to be summed over; and commas denote differentiation in the sense $u_{i,j} \equiv \dfrac{\partial u_i}{\partial x_j}$. Substitution of Eqs. (8) into Eqs. (7) yields the equilibrium relations in terms of the displacement components:

$$(1-2\nu)\, u_{i,jj} + u_{j,ji} = 0. \tag{9}$$

The solution of Eqs. (8) and (9) which satisfies the boundary conditions corresponding to the plane hydrostatic state of stress at infinity is elementary in form and is, in fact, exactly the associated plane-stress solution for this problem. The solution for the plane state of pure shear at infinity is, however, much more difficult to obtain. An approximation to this solution which is good at large distances from the plane $z = 0$ is the corresponding plane-strain solution. The difference between this approximate solution and the exact solution will be called the solution to the residual problem and is the subject of the major part of this paper. The Papkovich stress functions are used to obtain this solution; for the sake of uniformity the aforementioned well-known plane-stress and plane-strain solutions will also be established by means of these stress functions.

C. Papkovich Stress Functions

The general solution of Eqs. (9) can be expressed in the form

$$2\mu u_i = (\Phi + x_j\Psi_j)_{,i} - 2\alpha\Psi_i, \tag{10}$$

where the Papkovich stress functions Φ, Ψ_1, Ψ_2, and Ψ_3 are harmonic, i.e.,

$$\Phi_{,jj} = \Psi_{i,jj} = 0, \tag{11}$$

and α is defined by

$$\alpha \equiv 2(1 - \nu). \tag{12}$$

The substitution of Eqs. (10) into Eqs. (8) yields

$$\tau_{ij} = \Phi_{,ij} - (1 - 2\nu)(\Psi_{i,j} + \Psi_{j,i}) + x_k \Psi_{k,ij} - 2\nu\delta_{ij}\Psi_{k,k}, \tag{13}$$

so that the boundary conditions on the stresses can be expressed in terms of the stress functions. The reason for introducing the Papkovich stress functions is that the original nine coupled partial differential Eqs. (8) and (9) in the nine dependent variables u_i and τ_{ij} are converted into four standard Laplace's equations (11). Since much is known about the solutions to Laplace's equation, a great deal has been accomplished toward systematizing the solution of the problem. On the other hand, the boundary conditions, which were simple before [Eqs. (2), (3), and (5), for example], now become coupled partial differential equations when expressed in terms of the stress functions [see Eqs. (13)].

Referred to the cylindrical coordinates Eqs. (10) become

$$2\mu u_r = \frac{\partial}{\partial r}(\Phi + r\Psi_1 \cos\theta + r\Psi_2 \sin\theta + z\Psi_3) - 2\alpha(\Psi_1 \cos\theta + \Psi_2 \sin\theta);$$

$$2\mu u_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}(\Phi + r\Psi_1 \cos\theta + r\Psi_2 \sin\theta + z\Psi_3) - 2\alpha(-\Psi_1 \sin\theta + \Psi_2 \cos\theta);$$

$$2\mu u_z = \frac{\partial}{\partial z}(\Phi + r\Psi_1 \cos\theta + r\Psi_2 \sin\theta + z\Psi_3) - 2\alpha\Psi_3. \tag{14}$$

In cylindrical coordinates the boundary conditions for the problem of the plane hydrostatic state of stress at infinity are, by Eqs. (2), (3), and (4),

$$\tau^H_{rr} = \tau^H_{rz} = \tau^H_{r\theta} = 0 \quad \text{on } r = a,\ z \geq 0;$$

$$\tau^H_{zz} = \tau^H_{\theta z} = \tau^H_{rz} = 0 \quad \text{on } z = 0,\ r \geq a;$$

$$\left.\begin{aligned} &\tau^H_{rr} = \tau^H_{\theta\theta} = \tau_H; \quad \tau^H_{r\theta} = 0; \\ &\tau^H_{zz} = \tau^H_{\theta z} = \tau^H_{rz} = 0 \end{aligned}\right\} \text{at } r = \infty,\ z \geq 0, \tag{15}$$

where the superscript H has been used to identify this solution. Because of the geometry of the body and the nature of the boundary conditions (15), it is obvious that the problem is axially symmetric. For such problems, it has been proven[12]* that Ψ_1 and Ψ_2 may be set equal to zero, and Φ and Ψ_3 taken to be independent of θ without loss of completeness. Therefore, for the hydrostatic problem we will take

$$\begin{aligned}
\Phi(r, \theta, z) &= a^2 \tau_H \phi^H(\rho, \zeta); \\
\Psi_1(r, \theta, z) &= \Psi_2(r, \theta, z) = 0; \\
\Psi_3(r, \theta, z) &= a \tau_H \psi^H(\rho, \zeta),
\end{aligned} \tag{16}$$

where the dimensionless coordinates

$$\rho \equiv r/a; \qquad \zeta \equiv z/a \tag{17}$$

have been introduced, and ϕ^H and ψ^H are also dimensionless. The solution to Eqs. (11) and (16) for the boundary conditions derived from Eqs. (15), (13), and (16) is readily found to be

$$\phi^H = \frac{1-\nu}{1+\nu}\left(\frac{\rho^2}{2} - \zeta^2\right) + \log \rho; \qquad \psi^H = -\frac{\zeta}{1+\nu}. \tag{18}$$

The displacements and stresses corresponding to Eqs. (18) are, by Eqs. (14), (13), and (16),

$$\begin{aligned}
\frac{2\mu u_r^H}{a\tau_H} &= \frac{\partial}{\partial \rho}(\phi^H + \zeta\psi^H) = \frac{1-\nu}{1+\nu}\rho + \frac{1}{\rho}; \\
\frac{2\mu u_z^H}{a\tau_H} &= \frac{\partial}{\partial \zeta}(\phi^H + \zeta\psi^H) - 2\alpha\psi^H = -\frac{2\nu}{1+\nu}\zeta; \\
\frac{\tau_{rr}^H}{\tau_H} &= \frac{\partial^2}{\partial \rho^2}(\phi^H + \zeta\psi^H) - 2\nu\frac{\partial \psi^H}{\partial \zeta} = 1 - \frac{1}{\rho^2}; \\
\frac{\tau_{\theta\theta}^H}{\tau_H} &= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\phi^H + \zeta\psi^H) - 2\nu\frac{\partial \psi^H}{\partial \zeta} = 1 + \frac{1}{\rho^2}; \\
\frac{\tau_{zz}^H}{\tau_H} &= \frac{\partial^2}{\partial \zeta^2}(\phi^H + \zeta\psi^H) - 2(2-\nu)\frac{\partial \psi^H}{\partial \zeta} = 0; \\
\frac{\tau_{rz}^H}{\tau_H} &= \frac{\partial^2}{\partial \rho \partial \zeta}(\phi^H + \zeta\psi^H) - \alpha\frac{\partial \psi^H}{\partial \rho} = 0; \qquad u_\theta^H = \tau_{\theta z}^H = \tau_{r\theta}^H = 0.
\end{aligned} \tag{19}$$

*Superscript numbers in the text designate references listed at the end of the report.

In cylindrical coordinates the boundary conditions for the problem of the plane state of pure shear at infinity are, from Eqs. (2), (3), and (5),

$$\tau^S_{rr} = \tau^S_{rz} = \tau^S_{r\theta} = 0 \qquad \text{on } r = a,\ z \geq 0;$$

$$\tau^S_{zz} = \tau^S_{\theta z} = \tau^S_{rz} = 0 \qquad \text{on } z = 0,\ r \geq a;$$

$$\left.\begin{array}{l} \tau^S_{rr} = -\tau^S_{\theta\theta} = \tau_S \cos 2\theta; \quad \tau^S_{r\theta} = -\tau_S \sin 2\theta; \\ \tau^S_{zz} = \tau^S_{\theta z} = \tau^S_{rz} = 0 \end{array}\right\} \text{at } r = \infty,\ z \geq 0, \tag{20}$$

where the superscript S has been used to identify this solution. Since θ appears explicitly in these boundary conditions, it is clear that the problem is not axially symmetric. On the other hand, the angular variation is quite simple; u_r and u_z are proportional to $\cos 2\theta$ while u_θ is proportional to $\sin 2\theta$. Previous attempts to solve this problem have essentially involved taking Ψ_1 and Ψ_2 equal to zero, and Φ and Ψ_3 proportional to $\cos 2\theta$, proceeding along lines analogous to the axially symmetric case. This does indeed produce the desired θ variation, but it can be shown by a counter-example, namely, the plane-strain solution to be discussed below, that this procedure lacks sufficient generality. Instead, the stress function Ψ_1 and Ψ_2 must be combined in such a way as to produce the correct θ dependence. On inspection of Eqs. (14) it is seen that the proper angular variation will be achieved if

$$\Phi(r, \theta, z) = a^2\tau_S\phi^S(\rho, \zeta) \cos 2\theta;$$

$$\Psi_1(r, \theta, z) = a\tau_S\chi^S(\rho, \zeta) \cos \theta;$$

$$\Psi_2(r, \theta, z) = -a\tau_S\chi^S(\rho, \zeta) \sin \theta;$$

$$\Psi_3(r, \theta, z) = a\tau_S\psi^S(\rho, \zeta) \cos 2\theta. \tag{21}$$

Equations (14) then become

$$\frac{2\mu u_r^S}{a\tau_S} = \left[\frac{\partial}{\partial\rho}(\phi^S + \rho\chi^S + \zeta\psi^S) - 2\alpha\chi^S\right] \cos 2\theta;$$

$$\frac{2\mu u_\theta^S}{a\tau_S} = \left[-\frac{2}{\rho}(\phi^S + \rho\chi^S + \zeta\psi^S) + 2\alpha\chi^S\right] \sin 2\theta;$$

$$\frac{2\mu u_z^S}{a\tau_S} = \left[\frac{\partial}{\partial\zeta}(\phi^S + \rho\chi^S + \zeta\psi^S) - 2\alpha\psi^S\right] \cos 2\theta. \tag{22}$$

The stresses are expressed in terms of these stress functions by substituting Eqs. (22) into the stress-displacement relations in cylindrical coordinates; this yields

$$\frac{\tau^S_{rr}}{\tau_S} = \left[\frac{\partial^2}{\partial\rho^2}(\phi^S + \rho\chi^S + \zeta\psi^S) + (\alpha - 2)\left(\frac{\partial\psi^S}{\partial\zeta} - \frac{\chi^S}{\rho}\right) - (\alpha + 2)\frac{\partial\chi^S}{\partial\rho}\right]\cos 2\theta;$$

$$\frac{\tau^S_{\theta\theta}}{\tau_S} = \left[\left(\frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{4}{\rho^2}\right)(\phi^S + \rho\chi^S + \zeta\psi^S) + (\alpha - 2)\left(\frac{\partial\psi^S}{\partial\zeta} + \frac{\partial\chi^S}{\partial\rho}\right) + (\alpha + 2)\frac{\chi^S}{\rho}\right]\cos 2\theta;$$

$$\frac{\tau^S_{zz}}{\tau_S} = \left[\frac{\partial^2}{\partial\zeta^2}(\phi^S + \rho\chi^S + \zeta\psi^S) + (\alpha - 2)\left(\frac{\partial\chi^S}{\partial\rho} - \frac{\chi^S}{\rho}\right) - (\alpha + 2)\frac{\partial\psi^S}{\partial\zeta}\right]\cos 2\theta;$$

$$\frac{\tau^S_{r\theta}}{\tau_S} = \left\{-2\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}(\phi^S + \rho\chi^S + \zeta\psi^S)\right] + \alpha\left(\frac{\partial\chi^S}{\partial\rho} + \frac{\chi^S}{\rho}\right)\right\}\sin 2\theta;$$

$$\frac{\tau^S_{\theta z}}{\tau_S} = \left[-\frac{2}{\rho}\frac{\partial}{\partial\zeta}(\phi^S + \rho\chi^S + \zeta\psi^S) + \alpha\frac{\partial\chi^S}{\partial\zeta} + 2\alpha\frac{\psi^S}{\rho}\right]\sin 2\theta;$$

$$\frac{\tau^S_{rz}}{\tau_S} = \left[\frac{\partial^2}{\partial\rho\partial\zeta}(\phi^S + \rho\chi^S + \zeta\psi^S) - \alpha\frac{\partial\chi^S}{\partial\zeta} - \alpha\frac{\partial\psi^S}{\partial\rho}\right]\cos 2\theta. \tag{23}$$

By means of Eqs. (21) and (11) we find that the differential equations which the functions ϕ^S, χ^S, and ψ^S must satisfy are

$$\nabla_2^2\phi^S = \nabla_1^2\chi^S = \nabla_2^2\psi^S = 0, \tag{24}$$

where

$$\nabla_m^2 F \equiv \frac{\partial^2 F}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial\rho} + \frac{\partial^2 F}{\partial\zeta^2} - \frac{m^2}{\rho^2}F. \tag{25}$$

Thus the solution of the problem for the plane state of pure shear at infinity will be obtained if we can find ϕ^S, χ^S, and ψ^S such that Eqs. (24) are satisfied subject to the boundary conditions derived from Eqs. (20) and (23).

To simplify the problem somewhat, we will work with the residual problem created by subtracting the plane-strain solution* associated with the pure shear problem from the exact solution to the pure shear problem. To this end, consider

$$\phi^P = \frac{1}{2}\left(\rho^2 + \frac{1}{\rho^2}\right); \qquad \chi^P = -1/\rho; \qquad \psi^P = 0. \tag{26}$$

It is easily verified that Eqs. (26) satisfy Eqs. (24). The substitution of Eqs. (26) into Eqs. (22) and (23) yields

*The associated plane-strain solution satisfies the field equations and the boundary conditions on r = a, but, in general, not all of the boundary conditions on z = 0.

$$\frac{2\mu u_r^P}{a\tau_S} = \left(\rho + \frac{2\alpha}{\rho} - \frac{1}{\rho^3}\right)\cos 2\theta;$$

$$\frac{2\mu u_\theta^P}{a\tau_S} = -\left[\rho + \frac{2(\alpha - 1)}{\rho} + \frac{1}{\rho^3}\right]\sin 2\theta;$$

$$\frac{\tau_{rr}^P}{\tau_S} = \left(1 - \frac{4}{\rho^2} + \frac{3}{\rho^4}\right)\cos 2\theta;$$

$$\frac{\tau_{\theta\theta}^P}{\tau_S} = -\left(1 + \frac{3}{\rho^4}\right)\cos 2\theta;$$

$$\frac{\tau_{zz}^P}{\tau_S} = \frac{2(\alpha - 2)}{\rho^2}\cos 2\theta;$$

$$\frac{\tau_{r\theta}^P}{\tau_S} = \left(-1 - \frac{2}{\rho^2} + \frac{3}{\rho^4}\right)\sin 2\theta;$$

$$u_z^P = \tau_{\theta z}^P = \tau_{rz}^P = 0. \tag{27}$$

Inspection of Eqs. (27) shows that this is a plane-strain solution, and comparison with Eqs. (20) reveals that it is the plane-strain solution associated with the problem under discussion. Indeed, we see that Eqs. (27) satisfy all the boundary conditions (20) except* $\tau_{zz}^S = 0$ on $z = 0$ since $\tau_{zz}^P = 2(\alpha - 2)\,\tau_S\rho^{-2}\cos 2\theta$ for all values of z.

Consider now the residual problem, identified by a superscript R, obtained by subtracting the plane-strain state given by Eqs. (27) from the true solution, i.e., let

$$u_r^R \equiv u_r^S - u_r^P, \ \ldots, \ \ldots;$$

$$\tau_{rr}^R \equiv \tau_{rr}^S - \tau_{rr}^P, \ \ldots, \ \ldots, \ \ldots, \ \ldots;$$

$$\phi^R \equiv \phi^S - \phi^P; \qquad \chi^R \equiv \chi^S - \chi^P; \qquad \psi^R \equiv \psi^S - \psi^P. \tag{28}$$

We conclude that ϕ^R, χ^R, and ψ^R must satisfy Eqs. (24) subject to the boundary conditions

*The plane-strain solution satisfies this boundary condition also, and is therefore the exact solution to the pure shear problem, if $\alpha = 2$, i.e., by Eq. (12), if Poisson's ratio is zero.

$$\tau_{rr}^R = \tau_{rz}^R = \tau_{r\theta}^R = 0 \qquad \text{on } r = a,\ z \geq 0; \tag{29}$$

$$\tau_{zz}^R = \frac{2(2-\alpha)}{\rho^2}\cos 2\theta; \qquad \tau_{\theta z}^R = \tau_{rz}^R = 0 \qquad \text{on } z = 0,\ r \geq a; \tag{30}$$

$$\tau_{rr}^R = \tau_{\theta\theta}^R = \tau_{zz}^R = \tau_{r\theta}^R = \tau_{\theta z}^R = \tau_{rz}^R = 0 \qquad \text{at } r = \infty,\ z \geq 0, \tag{31}$$

which can be expressed in terms of the stress functions through Eqs. (23). The solution to this residual problem added to the plane-strain solution (27) is then the exact solution to the pure shear problem.

D. Construction of the Stress Functions for the Residual Problem

Rather than superimposing the solutions of the differential equations in a trial-and-error fashion in the hope of fortuitously satisfying the boundary conditions, it would seem preferable to use the powerful tools of integral transform methods to solve the problem systematically. This approach seems particularly encouraging in that parts of the solutions to Eqs. (24) obtained by separation of variables are common transform kernels. Although it turns out that integral transformations are not completely applicable to the solution of this type of problem, some of their properties can still be used to advantage in dealing with the boundary conditions.

It will be convenient to consider separately the mechanism for satisfying the boundary conditions on each of the two boundaries. To this end let

$$\phi^R = \phi_1 + \phi_2; \qquad \chi^R = \chi_1 + \chi_2; \qquad \psi^R = \psi_1 + \psi_2, \tag{32}$$

where the subscript 1 identifies the parts of the stress functions constructed to handle the boundary conditions (29) on the cylindrical surface and the subscript 2 identifies the parts constructed to handle the boundary conditions (30) on the plane surface.

1. Boundary Conditions on the Cylindrical Surface $\rho = 1$

As ζ is the independent variable on the boundary $\rho = 1$, examination of Eqs. (24) suggests the use of Fourier transforms with respect to this variable. If the Fourier cosine transform is taken of ϕ_1 and χ_1, and the Fourier sine transform of ψ_1, this will insure that their contributions to τ_{rz}^R and $\tau_{\theta z}^R$ will be odd functions of ζ and hence zero at $\zeta = 0$. Let

$$\hat{\phi}_1(\rho,\gamma) = \int_0^\infty \phi_1(\rho,\zeta)\cos\gamma\zeta\, d\zeta;$$

$$\hat{\chi}_1(\rho,\gamma) = \int_0^\infty \chi_1(\rho,\zeta)\cos\gamma\zeta\, d\zeta;$$

$$\hat{\psi}_1(\rho,\gamma) = \int_0^\infty \psi_1(\rho,\zeta)\sin\gamma\zeta\, d\zeta. \tag{33}$$

Applying these transformations to Eqs. (24) we are led to familiar differential equations in ρ which have the solutions

$$\hat{\phi}_1 = M_1(\gamma)\, K_2(\gamma\rho); \qquad \hat{\chi}_1 = M_2(\gamma)\, K_1(\gamma\rho); \qquad \hat{\psi}_1 = M_3(\gamma)\, K_2(\gamma\rho), \tag{34}$$

where K_n are the modified Bessel functions of the second kind[13] and M_1, M_2, M_3 are arbitrary functions. The modified Bessel functions of the first kind, $I_n(\gamma\rho)$, are also solutions, but have been discarded since they are unbounded as $\rho \to \infty$. Unfortunately, however, the boundary conditions on ϕ_1, χ_1, and ψ_1 deduced from Eqs. (23) and (29) cannot be transformed because of the $\zeta\psi$ terms which do not transform in terms of $\hat{\psi}_1$.

In general, integral transforms are used for two reasons: (a) to simplify the solution of differential equations by reducing the number of independent variables, and (b) to easily incorporate the boundary conditions into the solution. We have been able to take advantage of (a); by using the Fourier transform inversion theorem, and Eqs. (33) and (34), we find that

$$\phi_1 = \frac{2}{\pi}\int_0^\infty M_1(\gamma)\; K_2(\gamma\rho)\; \cos\gamma\zeta\; d\gamma;$$

$$\chi_1 = \frac{2}{\pi}\int_0^\infty M_2(\gamma)\; K_1(\gamma\rho)\; \cos\gamma\zeta\; d\gamma;$$

$$\psi_1 = \frac{2}{\pi}\int_0^\infty M_3(\gamma)\; K_2(\gamma\rho)\; \sin\gamma\zeta\; d\gamma. \tag{35}$$

However, these solutions are rather obvious because of the wealth of information available on the solutions of Laplace's equation. On the other hand, we have not been able to take advantage of (b), which is regrettable, as this was the primary reason for using transform methods here. This is not surprising though, as it is well-known that the use of stress functions simplifies the governing differential equations but complicates the boundary conditions.

It would seem advisable not to use transforms to solve the differential equations, but rather to use the available extensive information regarding the solutions to Eqs. (24) to make the boundary conditions amenable to transform techniques. An examination of Eqs. (23) reveals that the troublesome $\zeta\psi$ terms always occur in the combination $\phi + \rho\chi + \zeta\psi$; moreover, ϕ occurs only in this combination, never by itself. Can we therefore add something to ϕ which is a solution to the first of Eqs. (24) and which will give $\phi_1 + \rho\chi_1 + \zeta\psi_1$ the appearance of an inverse transform? At the moment, from Eqs. (35),

$$\phi_1 + \rho\chi_1 + \zeta\psi_1 = \frac{2}{\pi}\int_0^{\infty} \{[M_1(\gamma)\ K_2(\gamma\rho) + M_2(\gamma)\ \rho K_1(\gamma\rho)] \cos\gamma\zeta$$

$$+ M_3(\gamma)\ K_2(\gamma\rho)\ \zeta\ \sin\gamma\zeta\}\ d\gamma. \tag{36}$$

Indeed, there are solutions to Eqs. (24) which are not obtainable by separation of variables or transform methods; these can be found by trial and error using the differentiation and recursion relations for the Bessel functions or by noting that if f is harmonic, then $\vec{x} \cdot \nabla f$ is also harmonic. Therefore, since $K_2(\gamma\rho) \cos\gamma\zeta \cos 2\theta$ is harmonic,

$$\left(\rho\frac{\partial}{\partial\rho} + \zeta\frac{\partial}{\partial\zeta}\right)[K_2(\gamma\rho) \cos\gamma\zeta \cos 2\theta] = [\gamma\rho K_2'(\gamma\rho) \cos\gamma\zeta - \gamma\zeta K_2(\gamma\rho) \sin\gamma\zeta] \cos 2\theta, \tag{37}$$

where

$$K_2'(x) \equiv \frac{dK_2(x)}{dx},$$

is also harmonic; moreover, it is an even function of ζ. Instead of taking ϕ_1 as in Eqs. (35), let us take it in the form

$$\phi_1 = \frac{2}{\pi}\int_0^{\infty} \{M_1(\gamma)\ K_2(\gamma\rho) \cos\gamma\zeta + M_3(\gamma)[\rho K_2'(\gamma\rho) \cos\gamma\zeta - \zeta K_2(\gamma\rho) \sin\gamma\zeta]\}\ d\gamma. \tag{38}$$

Then Eq. (36) will be replaced by

$$\phi_1 + \rho\chi_1 + \zeta\psi_1 = \frac{2}{\pi}\int_0^{\infty} [M_1(\gamma)\ K_2(\gamma\rho) + M_2(\gamma)\ \rho K_1(\gamma\rho) + M_3(\gamma)\ \rho K_2'(\gamma\rho)] \cos\gamma\zeta\ d\gamma, \tag{39}$$

which has the proper transform form.

Inspection of Eqs. (23) reveals that all the boundary conditions applied at $\rho = 1$ can be expressed in terms of $\phi_1 + \rho\chi_1 + \zeta\psi_1$, χ_1, and ψ_1, evaluated at $\rho = 1$, which all have the proper transform appearance now. The three boundary conditions (29) will then lead to equations of the form

$$F_1(\zeta) = \int_0^{\infty} f_1[M_1(\gamma), M_2(\gamma), M_3(\gamma), \gamma] \cos\gamma\zeta\ d\gamma;$$

$$F_2(\zeta) = \int_0^{\infty} f_2[M_1(\gamma), M_2(\gamma), M_3(\gamma), \gamma] \sin\gamma\zeta\ d\gamma;$$

$$F_3(\zeta) = \int_0^{\infty} f_3[M_1(\gamma), M_2(\gamma), M_3(\gamma), \gamma] \cos\gamma\zeta\ d\gamma, \tag{40}$$

where the functions F_1, F_2, F_3 are derived from the applied loading and the functions ϕ_2, χ_2, and ψ_2 used to handle the boundary conditions on the bounding plane, and f_1, f_2, and f_3 are linear combinations of M_1, M_2, and M_3. From the Fourier transform inversion theorem we have*

$$f(\eta) = \frac{2}{\pi}\int_0^\infty\int_0^\infty f(\gamma) \begin{matrix}\cos\gamma\zeta\,\cos\eta\zeta\\ \sin\gamma\zeta\,\sin\eta\zeta\end{matrix}\, d\gamma\, d\zeta. \tag{41}$$

Therefore, multiplying each of Eqs. (40) by the appropriate trigonometric function and integrating with respect to ζ from zero to infinity will remove f_1, f_2, and f_3 from under the integral signs and result in three simultaneous algebraic equations for M_1, M_2, and M_3 in terms of the Fourier transforms of F_1, F_2, and F_3.

2. Boundary Conditions on the Plane Surface $\zeta = 0$

Since ϕ_1, χ_1, and ψ_1 have been constructed so that their contributions to the stresses $\tau^R_{\theta z}$ and τ^R_{rz} are zero at $\zeta = 0$, the functions ϕ_2, χ_2, and ψ_2 should also be constructed so that their contributions to these stresses are zero at $\zeta = 0$. It is apparent from (27) that this will be accomplished if

$$\frac{\partial\phi_2}{\partial\zeta} = (\alpha - 1)\,\psi_2; \qquad \chi_2 = 0. \tag{42}$$

Perhaps the most obvious means of satisfying the remaining boundary condition, the condition on τ^R_{zz} of Eqs. (30), is to transform the differential equations (24) for ϕ and ψ by the Hankel transform of order two. This leads to very simple differential equations in ζ which have as their solutions $e^{+\gamma\zeta}$ and $e^{-\gamma\zeta}$. Inversion then produces the stress functions;** by means of Eqs. (42),

$$\phi_2 = (1-\alpha)\int_0^\infty \gamma^{-1}M_4(\gamma)\,J_2(\gamma\rho)\,e^{-\gamma\zeta}\,d\gamma; \qquad \chi_2 = 0;$$

$$\psi_2 = \int_0^\infty M_4(\gamma)\,J_2(\gamma\rho)\,e^{-\gamma\zeta}\,d\gamma, \tag{43}$$

where J_2 is the Bessel function of the first kind of order two. The boundary condition then becomes

$$\frac{2(2-\alpha)}{\rho^2} + F_4(\rho) = \int_0^\infty \gamma M_4(\gamma)\,J_2(\gamma\rho)\,d\gamma, \quad \rho \geq 1, \tag{44}$$

*See Ref. 14, p. 15.

**The solution corresponding to $e^{+\gamma\zeta}$ can be dropped because of its behavior as $\zeta \to \infty$.

where F_4 results from the functions ϕ_1, χ_1, and ψ_1 previously discussed. The Hankel transform inversion theorem can be written*

$$f(\eta) = \int_0^\infty \int_0^\infty f(\gamma)\, \gamma\rho J_m(\gamma\rho)\, J_m(\eta\rho)\, d\gamma d\rho. \tag{45}$$

In order to apply this we must multiply Eq. (44) by $\rho J_2(\eta\rho)$ and integrate with respect to ρ from zero to infinity. However, this is not a valid procedure since no boundary condition is prescribed for $0 \leq \rho < 1$ where there is no regional boundary. It follows that the Hankel transform cannot be used; what is needed is a transformation having a range of integration from one to infinity. We will therefore follow a procedure similar to that used by Blenkarn and Wilhoit.[10]

Consider Weber's Integral Theorem,** according to which a properly behaved function f can be given the following representation:

$$f(\eta)\left[J_m^2(\eta) + Y_m^2(\eta)\right] = \int_1^\infty \int_0^\infty \gamma\rho f(\gamma)\, \Omega_m^0(\gamma,\rho)\, \Omega_m^0(\eta,\rho)\, d\gamma d\rho, \tag{46}$$

where

$$\Omega_m^0(\gamma,\rho) \equiv Y_m(\gamma)\, J_m(\gamma\rho) - J_m(\gamma)\, Y_m(\gamma\rho) \tag{47}$$

and Y_m is the Bessel function of the second kind of order m. Note that the integration with respect to ρ is from one to infinity and that Eq. (46) has the form of an inversion theorem for a transform with kernel Ω_m^0. Since in this problem $\rho \geq 1$, the function $Y_m(\gamma\rho)$ is quite admissible as part of the solution, in contrast to problems conveniently solved by use of Hankel transforms where it would be unsuitable because of its infinite value at $\rho = 0$.

A solution to the residual problem discussed in this report was obtained using the transformation kernel Ω_2^0. It was discovered, however, that, although all integrals involved in the final expressions for the stresses were theoretically convergent, the difficulties encountered in their numerical evaluation were apparently prohibitive. The problem was then successfully solved by a slightly different transformation, which can be summarized by the inversion relation†

$$f(\eta)\left\{\left[J'_m(\eta)\right]^2 + \left[Y'_m(\eta)\right]^2\right\} = \int_1^\infty \int_0^\infty \gamma\rho f(\gamma)\, \Omega_m(\gamma,\rho)\, \Omega_m(\eta,\rho)\, d\gamma d\rho. \tag{48}$$

*See Ref. 14, p. 16.

**See Ref. 13, pp. 468-470, for details.

†See Appendix C for a proof of this relation.

In Eq. (48)

$$\Omega_m(\gamma,\rho) \equiv Y'_m(\gamma)\, J_m(\gamma\rho) - J'_m(\gamma)\, Y_m(\gamma\rho); \qquad Y'_m(\gamma) \equiv \frac{dY_m(\gamma)}{d\gamma}; \qquad J'_m(\gamma) \equiv \frac{dJ_m(\gamma)}{d\gamma}; \tag{49}$$

here m = 2 is the appropriate value since $\Omega_2(\gamma,\rho)\, e^{-\gamma\zeta}$ is a solution to the first and third of Eqs. (24). Therefore, in place of Eqs. (43) we will take the stress functions ϕ_2, χ_2, and ψ_2 in the form

$$\phi_2 = (1-\alpha)\int_0^\infty \gamma^{-1} M_4(\gamma)\, \Omega_2(\gamma,\rho)\, e^{-\gamma\zeta}\, d\gamma; \qquad \chi_2 = 0;$$

$$\psi_2 = \int_0^\infty M_4(\gamma)\, \Omega_2(\gamma,\rho)\, e^{-\gamma\zeta}\, d\gamma. \tag{50}$$

The boundary condition then becomes, rather than Eq. (44),

$$\frac{2(2-\alpha)}{\rho^2} + F_4(\rho) = \int_0^\infty \gamma M_4(\gamma)\, \Omega_2(\gamma,\rho)\, d\gamma; \qquad \rho \geq 1. \tag{51}$$

Applying the inversion relation (48) to Eq. (51) gives

$$M_4(\eta)\left\{\left[J'_2(\eta)\right]^2 + \left[Y'_2(\eta)\right]^2\right\} = \int_1^\infty \rho\left[\frac{2(2-\alpha)}{\rho^2} + F_4(\rho)\right] \Omega_2(\eta,\rho)\, d\rho, \tag{52}$$

so that the undetermined function M_4 has been removed from under the integration sign. Using the integral transformation implied by Eq. (48) rather than that implied by Eq. (46), we obtain integrals for the stresses which are very similar in appearance and have the same theoretical rates of convergence, but which are more amenable to numerical evaluation.

E. Solution of the Residual Problem

In accordance with the reasoning of the previous section, we take the stress functions in the form

$$\phi^R(\rho,\zeta) = \int_0^\infty \Bigg\{\left[A(\gamma) - \frac{2}{\alpha}C(\gamma)\right] K_2(\gamma\rho)\cos\gamma\zeta + \left[B(\gamma) - \frac{1}{\alpha}C(\gamma)\right]\left[\gamma\rho K'_2(\gamma\rho)\cos\gamma\zeta - \gamma\zeta K_2(\gamma\rho)\sin\gamma\zeta\right]$$

$$+ (1-\alpha)\,\gamma^2 D(\gamma)\, \Omega_2(\gamma,\rho)\, e^{-\gamma\zeta}\Bigg\} d\gamma;$$

$$\chi^R(\rho,\zeta) = \frac{1}{\alpha}\int_0^\infty C(\gamma)\left[\frac{2}{\rho}K_2(\gamma\rho) + \gamma K'_2(\gamma\rho)\right]\cos\gamma\zeta\, d\gamma;$$

$$\psi^R(\rho,\zeta) = \int_0^\infty \left\{\left[B(\gamma) - \frac{1}{\alpha}C(\gamma)\right]\gamma K_2(\gamma\rho)\sin\gamma\zeta + \gamma^3 D(\gamma)\, \Omega_2(\gamma,\rho)\, e^{-\gamma\zeta}\right\} d\gamma. \tag{53}$$

The undetermined functions A, B, C, and D have been arranged so as to simplify subsequent expressions. Note that the recurrence relation for the K_n Bessel functions has been used to express $K_1(\gamma\rho)$ in terms of $K_2(\gamma\rho)$ and $K_2'(\gamma\rho)$ in the equation for χ^R. The stress functions (53) satisfy the differential equations (24); consequently the stress field derived from them will meet the equilibrium equations (7).

Substituting Eqs. (53) into the equations in (23) for τ_{rr}, $\tau_{r\theta}$, and τ_{rz}, evaluating the resulting expressions at $\rho = 1$, and applying the boundary conditions (29), we find

$$\int_0^\infty \{A(\gamma)[\gamma^2 + 4 - K(\gamma)] + B(\gamma)[(\alpha - 1)\gamma^2 - 4 + (\gamma^2+4)K(\gamma)]$$

$$- 2C(\gamma)[\gamma^2 + 2 + K(\gamma)]\}\, K_2(\gamma)\cos\gamma\zeta\, d\gamma$$

$$= \frac{2}{\pi}\int_0^\infty \gamma D(\gamma)[4(\alpha - 1 - \gamma\zeta) + \gamma^2(\gamma\zeta - 1)]\, e^{-\gamma\zeta}\, d\gamma;$$

$$\int_0^\infty \{2A(\gamma)[1 - K(\gamma)] + 2B(\gamma)[-(\gamma^2+4) + K(\gamma)]$$

$$+ C(\gamma)[\gamma^2 + 4 + 2K(\gamma)]\}\, K_2(\gamma)\cos\gamma\zeta\, d\gamma = \frac{4}{\pi}\int_0^\infty \gamma D(\gamma)(\alpha - 1 - \gamma\zeta)\, e^{-\gamma\zeta}\, d\gamma;$$

$$\int_0^\infty \{A(\gamma)K(\gamma) + B(\gamma)[\gamma^2 + 4 + \alpha K(\gamma)] - 2C(\gamma)[1 + K(\gamma)]\}\,\gamma K_2(\gamma)\sin\gamma\zeta\, d\gamma = 0, \tag{54}$$

where

$$K(\gamma) \equiv \gamma K_2'(\gamma)/K_2(\gamma) \tag{55}$$

and the relation*

$$\Omega_2(\gamma, 1) = 2/\pi\gamma \tag{56}$$

has been used. Upon multiplication of the first two of Eqs. (54) by $(2/\pi)\cos\eta\zeta$ and the third by $(2/\pi)\sin\eta\zeta$, integration with respect to ζ from zero to infinity, and use of Eq. (41), we obtain

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*See Ref. 15, p. 79, No. 28.

$$A(\eta)[\eta^2 + 4 - K(\eta)] + B(\eta)[(\alpha - 1)\eta^2 - 4 + (\eta^2+4)K(\eta)] - 2C(\eta)[\eta^2 + 2 + K(\eta)]$$

$$= \frac{8}{\pi^2 K_2(\eta)} \int_0^\infty D(\gamma)\left[2\alpha - (\eta^2+4)\, g\left(\frac{\eta}{\gamma}\right)\right] g\left(\frac{\eta}{\gamma}\right) d\gamma;$$

$$2A(\eta)[1 - K(\eta)] + 2B(\eta)[-(\eta^2+4) + K(\eta)] + C(\eta)[\eta^2 + 4 + 2K(\eta)]$$

$$= \frac{8}{\pi^2 K_2(\eta)} \int_0^\infty D(\gamma)\left[\alpha - 2g\left(\frac{\eta}{\gamma}\right)\right] g\left(\frac{\eta}{\gamma}\right) d\gamma;$$

$$A(\eta)\, K(\eta) + B(\eta)[\eta^2 + 4 + \alpha K(\eta)] - 2C(\eta)[1 + K(\eta)] = 0, \tag{57}$$

where

$$g(x) \equiv (1+x^2)^{-1} \tag{58}$$

The Fourier cosine transforms[16]

$$\int_0^\infty \gamma e^{-\gamma\zeta} \cos \eta\zeta \, d\zeta = g\left(\frac{\eta}{\gamma}\right);$$

$$\int_0^\infty \gamma(1+\gamma\zeta)\, e^{-\gamma\zeta} \cos \eta\zeta \, d\zeta = 2g^2\left(\frac{\eta}{\gamma}\right) \tag{59}$$

have been used in the derivation of Eqs. (57). Equations (57) are three simultaneous linear algebraic equations in A, B, and C; their solution, after a convenient change of the dummy variables, is

$$A(\gamma) = \frac{16\alpha}{\pi^2 K_2(\gamma)\,\Delta(\gamma)} [f_1(\gamma) + f_4(\gamma)] \int_0^\infty D(\xi)\, g\left(\frac{\gamma}{\xi}\right) d\xi + \frac{8}{\pi^2 K_2(\gamma)\,\Delta(\gamma)} \{-[\gamma^2 + 4 + \alpha K(\gamma)]\, f_3(\gamma)$$

$$+ 4\alpha f_2(\gamma)[K(\gamma) + 1]\} \int_0^\infty D(\xi)\, g^2\left(\frac{\gamma}{\xi}\right) d\xi;$$

$$B(\gamma) = \frac{16\alpha}{\pi^2 K_2(\gamma)\,\Delta(\gamma)} f_2(\gamma) \int_0^\infty D(\xi)\, g\left(\frac{\gamma}{\xi}\right) d\xi + \frac{8}{\pi^2 K_2(\gamma)\,\Delta(\gamma)} K(\gamma)\, f_3(\gamma) \int_0^\infty D(\xi)\, g^2\left(\frac{\gamma}{\xi}\right) d\xi;$$

$$C(\gamma) = \frac{8\alpha}{\pi^2 K_2(\gamma)\,\Delta(\gamma)} f_1(\gamma) \int_0^\infty D(\xi)\, g\left(\frac{\gamma}{\xi}\right) d\xi + \frac{16\alpha}{\pi^2 K_2(\gamma)\,\Delta(\gamma)} K(\gamma)\, f_2(\gamma) \int_0^\infty D(\xi)\, g^2\left(\frac{\gamma}{\xi}\right) d\xi, \tag{60}$$

where

$$f_1(\gamma) \equiv \gamma^2[\gamma^2 + 4 |- K^2(\gamma)] + 3\alpha K^2(\gamma);$$

$$f_2(\gamma) \equiv \gamma^2 - (\gamma^2 + 3)\ K(\gamma);$$

$$f_3(\gamma) \equiv (\gamma^2 + 2)(\gamma^2 + 6) - 2\gamma^2 K(\gamma);$$

$$f_4(\gamma) \equiv (\gamma^2 + 3)(\gamma^2 + 4) - \gamma^2 K(\gamma) - \alpha f_2(\gamma) + 3\alpha K(\gamma);$$

$$\Delta(\gamma) \equiv [\gamma^2 + 4 - K^2(\gamma)]\ f_3(\gamma) + \alpha[-4\gamma^2 + 8(\gamma^2 + 3)\ K(\gamma) - \gamma^2 K^2(\gamma) - 6K^3(\gamma)]. \tag{61}$$

Substituting Eqs. (53) into the equations in (23) for τ_{zz}, $\tau_{\theta z}$, and τ_{rz}, evaluating the resulting expressions at $\zeta = 0$, and referring to Eqs. (30), we see that the boundary conditions for $\tau^R_{\theta z}$ and τ^R_{rz} are automatically satisfied, while the boundary condition for τ^R_{zz} is equivalent to

$$\int_0^\infty \gamma^4 D(\gamma)\ \Omega_2(\gamma, \rho)\ d\gamma = \frac{2(2-\alpha)}{\rho^2} + \int_0^\infty \{[A(\gamma) + (\alpha + 2)\ B(\gamma) - 2C(\gamma)]\ K_2(\gamma\rho)$$
$$+ B(\gamma)\ \gamma\rho K_2'(\gamma\rho)\}\ \gamma^2\ d\gamma. \tag{62}$$

Multiply Eq. (62) by $\rho\Omega_2(\eta, \rho)$ and integrate with respect to ρ from one to infinity, and use Eq. (48) to obtain

$$D(\eta)\ \eta^3\{[J_2'(\eta)]^2 + [Y_2'(\eta)]^2\}$$
$$= \frac{8(2-\alpha)}{\pi\eta^3} - \frac{2}{\pi\eta^3}\int_0^\infty \Big\{A(\gamma)\ K(\gamma) + B(\gamma)[\gamma^2 + 4 + \alpha K(\gamma)] - 2C(\gamma)\ K(\gamma)$$
$$+ 2B(\gamma)\ K(\gamma)\ g\left(\frac{\gamma}{\eta}\right)\Big\}\ g\left(\frac{\gamma}{\eta}\right)\ \gamma^2 K_2(\gamma)\ d\gamma, \tag{63}$$

where use has been made of the integrals*

---

*These integrals can be found from the indefinite integral

$$\int \rho\,\Omega_2(\eta, \rho)\ K_2(\gamma\rho)\ d\rho = \frac{\rho}{\gamma^2 + \eta^2}\left[\Omega_2(\eta, \rho)\ \frac{\partial K_2(\gamma\rho)}{\partial\rho} - K_2(\gamma\rho)\ \frac{\partial\Omega_2(\eta, \rho)}{\partial\rho}\right],$$

which can be checked by differentiation (see Appendix A, Sect. 3). The second of Eqs. (64) is obtained by putting in the proper limits; the first is found from this result by multiplying by γ^2 and letting $\gamma \to 0$; the third is found by differentiating with respect to γ.

$$\int_1^\infty \rho\Omega_2(\eta,\rho)\left(\frac{1}{\rho^2}\right) d\rho = \frac{4}{\pi\eta^3};$$

$$\int_1^\infty \rho\Omega_2(\eta,\rho)\, K_2(\gamma\rho)\, d\rho = -\frac{2\gamma K_2'(\gamma)}{\pi\eta^3}\, g\left(\frac{\gamma}{\eta}\right);$$

$$\int_1^\infty \rho\Omega_2(\eta,\rho)\, \gamma\rho K_2'(\gamma\rho)\, d\rho = \frac{2}{\pi\eta^3}\left\{-(\gamma^2+4)\, K_2(\gamma) + 2\gamma K_2'(\gamma)\left[1 - g\left(\frac{\gamma}{\eta}\right)\right]\right\} g\left(\frac{\gamma}{\eta}\right). \tag{64}$$

The functions A, B, and C of Eqs. (60) can now be substituted into Eq. (63) to yield an integral equation in the function D:

$$D(\eta)\eta^6 Y(\eta) = \frac{8(2-\alpha)}{\pi} + \int_0^\infty D(\xi)\, L(\xi,\eta)\, d\xi, \tag{65}$$

where

$$Y(\eta) \equiv [J_2'(\eta)]^2 + [Y_2'(\eta)]^2 \tag{66}$$

and the symmetric kernel $L(\xi,\eta)$ is defined by

$$L(\xi,\eta) \equiv -\frac{32}{\pi^3}\int_0^\infty \frac{\gamma^2}{\Delta(\gamma)}\, g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right)$$

$$\left\{\alpha f_1(\gamma) + 2\alpha\left[g\left(\frac{\gamma}{\xi}\right) + g\left(\frac{\gamma}{\eta}\right)\right] K(\gamma)\, f_2(\gamma) + g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right) K^2(\gamma)\, f_3(\gamma)\right\} d\gamma. \tag{67}$$

The integral equation (65) was solved with the IBM-704 for Poisson's ratios of 1/4 and 1/2 (that is, $\alpha = 3/2$ and 1). Figures 2 and 3 are graphs of the $D(\eta)$ functions. Details of the calculations are presented in Part II.

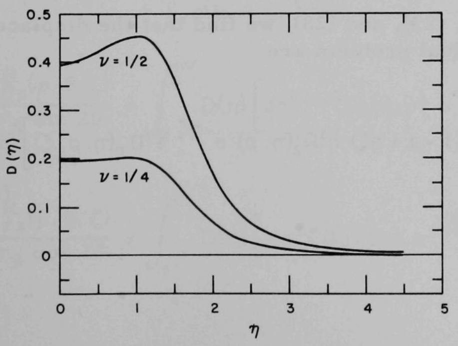

Fig. 2

The Solution $D(\eta)$ of the Integral Equation for Poisson's Ratios of 1/4 and 1/2

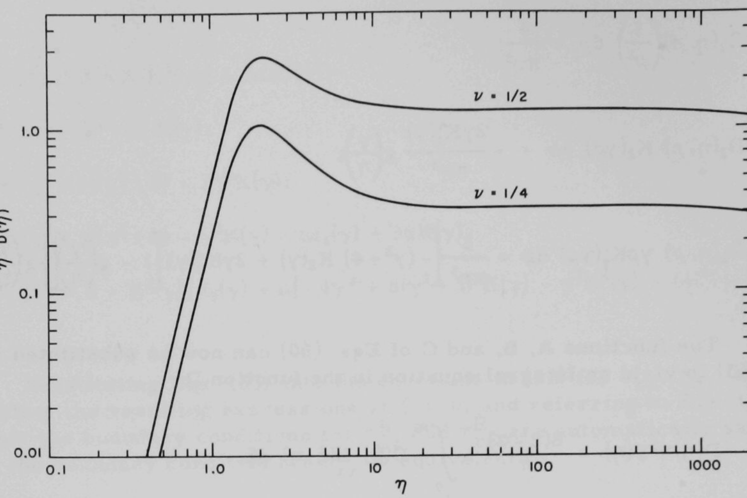

Fig. 3. Log-Log Plot of $\eta^4 D(\eta)$ vs. η for Poisson's Ratios of 1/4 and 1/2

F. Final Form of the Displacements and Stresses for the Residual Problem

Define the auxiliary notation

$$\Omega_2'(\gamma, \rho) \equiv Y_2'(\gamma)\, J_2'(\gamma\rho) - J_2'(\gamma)\, Y_2'(\gamma\rho);$$

$$G(\gamma, \rho) \equiv K_2(\gamma\rho)/K_2(\gamma); \qquad G'(\gamma, \rho) \equiv \rho K_2(\gamma\rho)/K_2'(\gamma). \tag{68}$$

The stress functions (53) are expressed in terms of the four functions A, B, C, and D. Since by Eqs. (60) we have the first three of these as functions of the fourth, we can express the stress functions, and thereby the displacements and stresses, in terms of D alone, where D is the solution of the integral equation (65).

Hence, by Eqs. (60), (53), (23), and (28), we find that the displacements and stresses for the residual problem are:

$$\frac{2\mu u_r^R(\rho, \theta, \zeta)}{a\tau_S \cos 2\theta} = \int_0^\infty D(\eta)\left[(1 - \alpha + \eta\zeta)\, \eta^3 \Omega_2'(\eta, \rho)\, e^{-\eta\zeta} + U_r(\eta, \rho, \zeta)\right] d\eta; \tag{69}$$

$$\frac{2\mu u_\theta^R(\rho,\theta,\zeta)}{a\tau_S \sin 2\theta} = \int_0^\infty D(\eta)\left[2(\alpha - 1 - \eta\zeta)\,\eta^2\rho^{-1}\Omega_2(\eta,\rho)\;e^{-\eta\zeta} + U_\theta(\eta,\rho,\zeta)\right]d\eta;$$

$$\frac{2\mu u_z^R(\rho,\theta,\zeta)}{a\tau_S \cos 2\theta} = \int_0^\infty D(\eta)\left[-(\alpha + \eta\zeta)\,\eta^3\Omega_2(\eta,\rho)\;e^{-\eta\zeta} + U_z(\eta,\rho,\zeta)\right]d\eta;$$

$$\frac{\tau_{rr}^R(\rho,\theta,\zeta)}{\tau_S \cos 2\theta} = \int_0^\infty D(\eta)\left[\left\{[4(1 - \alpha + \eta\zeta)\,\rho^{-2} + (1 - \eta\zeta)\,\eta^2]\;\Omega_2(\eta,\rho)\right.\right.$$

$$\left.\left. + (\alpha - 1 - \eta\zeta)\,\eta\rho^{-1}\Omega_2'(\eta,\rho)\right\}\eta^2\,e^{-\eta\zeta} + T_{rr}(\eta,\rho,\zeta)\right]d\eta;$$

$$\frac{\tau_{\theta\theta}^R(\rho,\theta,\zeta)}{\tau_S \cos 2\theta} = \int_0^\infty D(\eta)\left[\left\{[4(\alpha - 1 - \eta\zeta)\,\rho^{-2} + (2 - \alpha)\,\eta^2]\;\Omega_2(\eta,\rho)\right.\right.$$

$$\left.\left. + (1 - \alpha + \eta\zeta)\,\eta\rho^{-1}\Omega_2'(\eta,\rho)\right\}\eta^2\,e^{-\eta\zeta} + T_{\theta\theta}(\eta,\rho,\zeta)\right]d\eta;$$

$$\frac{\tau_{zz}^R(\rho,\theta,\zeta)}{\tau_S \cos 2\theta} = \int_0^\infty D(\eta)\left[(1 + \eta\zeta)\,\eta^4\Omega_2(\eta,\rho)\;e^{-\eta\zeta} + T_{zz}(\eta,\rho,\zeta)\right]d\eta;$$

$$\frac{\tau_{r\theta}^R(\rho,\theta,\zeta)}{\tau_S \sin 2\theta} = \int_0^\infty D(\eta)\left\{2(1 - \alpha + \eta\zeta)[\rho^{-2}\Omega_2(\eta,\rho) - \eta\rho^{-1}\Omega_2'(\eta,\rho)]\eta^2\,e^{-\eta\zeta}\right.$$

$$\left. + T_{r\theta}(\eta,\rho,\zeta)\right\}d\eta;$$

$$\frac{\tau_{\theta z}^R(\rho,\theta,\zeta)}{\tau_S \sin 2\theta} = \int_0^\infty D(\eta)\left[2\eta^4\rho^{-1}\zeta\Omega_2(\eta,\rho)\;e^{-\eta\zeta} + T_{\theta z}(\eta,\rho,\zeta)\right]d\eta;$$

$$\frac{\tau_{rz}^R(\rho,\theta,\zeta)}{\tau_S \cos 2\theta} = \int_0^\infty D(\eta)\left[-\eta^5\zeta\Omega_2'(\eta,\rho)\;e^{-\eta\zeta} + T_{rz}(\eta,\rho,\zeta)\right]d\eta; \qquad (69)\ \text{Contd.}$$

$$\frac{\tau_{DD}^{R}(\rho,\theta,\zeta)}{\tau_S\cos 2\theta} \equiv \frac{\tau_{rr}^{R}+\tau_{\theta\theta}^{R}+\tau_{zz}^{R}}{\tau_S\cos 2\theta}$$

$$= (4-\alpha)\int_0^\infty D(\eta)\left[\eta^4\Omega_2(\eta,\rho)\ e^{-\eta\zeta} + T_{DD}(\eta,\rho,\zeta)\right] d\eta, \tag{69 Contd.}$$

where

$$U_r(\eta,\rho,\zeta) \equiv \frac{8}{\pi^2}\int_0^\infty \frac{\cos\gamma\zeta}{\rho\Delta}\Big[2\alpha g\left(\frac{\gamma}{\eta}\right)\{G(\gamma,\rho)[(\gamma^2\rho^2+4)\ f_2 - 2f_1]$$

$$+ G'(\gamma,\rho)\ Kf_4\} + g^2\left(\frac{\gamma}{\eta}\right) K\{G(\gamma,\rho)[(\gamma^2\rho^2+4)\ f_3 - 8\alpha f_2]$$

$$+ G'(\gamma,\rho)[-(\gamma^2+4+\alpha K)\ f_3 + 4\alpha f_2]\}\Big]\ d\gamma;$$

$$U_\theta(\eta,\rho,\zeta) \equiv \frac{16}{\pi^2}\int_0^\infty \frac{\cos\gamma\zeta}{\rho\Delta}\Big[\alpha g\left(\frac{\gamma}{\eta}\right)\{-2G(\gamma,\rho)\ f_4 + G'(\gamma,\rho)\ K[f_1-2f_2]\}$$

$$+ g^2\left(\frac{\gamma}{\eta}\right)\{G(\gamma,\rho)[(\gamma^2+4+\alpha K)\ f_3 - 4\alpha f_2]$$

$$\pm G'(\gamma,\rho)\ K^2[-f_3 + 2\alpha f_2]\}\Big]\ d\gamma;$$

$$U_z(\eta,\rho,\zeta) \equiv \frac{8}{\pi^2}\int_0^\infty \frac{\gamma\sin\gamma\zeta}{\Delta}\Big[-2\alpha g\left(\frac{\gamma}{\eta}\right)\{G(\gamma,\rho)[f_4+2\alpha f_2] + G'(\gamma,\rho)\ Kf_2\}$$

$$+ g^2\left(\frac{\gamma}{\eta}\right)\{G(\gamma,\rho)[(\gamma^2+4-\alpha K)\ f_3 - 4\alpha f_2] - G'(\gamma,\rho)\ K^2 f_3\}\Big]\ d\gamma;$$

$$T_{rr}(\eta,\rho,\zeta) \equiv \frac{8}{\pi^2}\int_0^\infty \frac{\cos\gamma\zeta}{\rho^2\Delta}\Big\{2\alpha g\left(\frac{\gamma}{\eta}\right)\Big[G(\gamma,\rho)\{[(\alpha-1)\ \gamma^2\rho^2 - 4]\ f_2 + 2f_1$$

$$+ (\gamma^2\rho^2+4)\ f_4\} + G'(\gamma,\rho)\ K[(\gamma^2\rho^2+4)\ f_2 - 2f_1 - f_4]\Big]$$

$$+ g^2\left(\frac{\gamma}{\eta}\right)\Big[G(\gamma,\rho)\{-[(\gamma^2\rho^2+4)(\gamma^2+4+K) + 4\alpha K]\ f_3$$

$$+ 4\alpha(\gamma^2\rho^2+4+2K)\ f_2\} + G'(\gamma,\rho)\ K\{[(\gamma^2\rho^2+4+\alpha)\ K$$

$$+ \gamma^2 + 4]\ f_3 - 4\alpha(2K+1)\ f_2\}\Big]\Big\}\ d\gamma; \tag{70}$$

$$T_{\theta\theta}(\eta, \rho, \zeta) \equiv \frac{8}{\pi^2}\int_0^\infty \frac{\cos\gamma\zeta}{\rho^2\Delta}\left\{2\alpha g\left(\frac{\gamma}{\eta}\right)\left[G(\gamma, \rho)\{[(\alpha - 1)\gamma^2\rho^2 + 4]\, f_2 - 2f_1 - 4f_4\}\right.\right.$$

$$\left. + G'(\gamma, \rho)\, K[-4f_2 + 2f_1 + f_4]\right]$$

$$+ g^2\left(\frac{\gamma}{\eta}\right)\left[G(\gamma, \rho)\{[(\alpha - 1)\gamma^2\rho^2 K + 4(\gamma^2 + 4 + K + \alpha K)]\, f_3\right.$$

$$- 8\alpha(K + 2)\, f_2\} + G'(\alpha, \rho)\, K\{-(\gamma^2 + 4 + 4K + \alpha K)\, f_3$$

$$\left.\left. + 4\alpha(2K + 1)\, f_2\}\right]\right\} d\gamma;$$

$$T_{zz}(\eta, \rho, \zeta) \equiv \frac{8}{\pi^2}\int_0^\infty \frac{\gamma^2\cos\gamma\zeta}{\Delta}\left[-2\alpha g\left(\frac{\gamma}{\eta}\right)\{G(\gamma, \rho)[(\alpha + 2)\, f_2 + f_4]\right.$$

$$+ G'(\gamma, \rho)\, Kf_2\} + g^2\left(\frac{\gamma}{\eta}\right)\{G(\gamma, \rho)[(\gamma^2 + 4 - 2K)\, f_3$$

$$\left. - 4\alpha f_2] - G'(\gamma, \rho)\, K^2 f_3\}\right] d\gamma;$$

$$T_{r\theta}(\eta, \rho, \zeta) \equiv \frac{8}{\pi^2}\int_0^\infty \frac{\cos\gamma\zeta}{\rho^2\Delta}\left\{\alpha g\left(\frac{\gamma}{\eta}\right)\{G(\gamma, \rho)[-4(\gamma^2\rho^2 + 4)\, f_2\right.$$

$$+ (\gamma^2\rho^2 + 8)\, f_1 + 4f_4] + 2G'(\gamma, \rho)\, K[2f_2 - f_1 - 2f_4]\}$$

$$+ 2g^2\left(\frac{\gamma}{\eta}\right)\left[G(\gamma, \rho)\{-[(\gamma^2\rho^2 + 4 + \alpha)\, K + \gamma^2 + 4]\, f_3\right.$$

$$+ \alpha(\gamma^2\rho^2 K + 4 + 8K)\, f_2\} + G'(\gamma, \rho)\, K\{(\gamma^2 + 4$$

$$\left.\left. + K + \alpha K)\, f_3 - 2\alpha(2 + K)\, f_2\}\right]\right\} d\gamma;$$

$$T_{\theta z}(\eta, \rho, \zeta) \equiv \frac{8}{\pi^2}\int_0^\infty \frac{\gamma\sin\gamma\zeta}{\rho\Delta}\left[\alpha g\left(\frac{\gamma}{\eta}\right)\{4G(\gamma, \rho)[\alpha f_2 + f_4]\right.$$

$$+ G'(\gamma, \rho)\, K[4f_2 - f_1]\} + 2g^2\left(\frac{\gamma}{\eta}\right)\{G(\gamma, \rho)[-(\gamma^2 + 4)\, f_3$$

$$\left. + 4\alpha f_2] + G'(\gamma, \rho)\, K^2[f_3 - \alpha f_2]\}\right] d\gamma; \qquad (70)\ \text{Contd.}$$

$$T_{rz}(\eta, \rho, \zeta) \equiv \frac{8}{\pi^2} \int_0^\infty \frac{\gamma \sin \gamma\zeta}{\rho\Delta} \Big[2\alpha g\left(\frac{\gamma}{\eta}\right) \{G(\gamma, \rho)[-(\gamma^2\rho^2 + 4) f_2 + f_1]$$

$$- G'(\gamma, \rho) K[\alpha f_2 + f_4]\} + g^2\left(\frac{\gamma}{\eta}\right) K\{G(\gamma, \rho)[-(\gamma^2\rho^2 + 4) f_3$$

$$+ 4\alpha f_2] + G'(\gamma, \rho)[(\gamma^2 + 4) f_3 - 4\alpha f_2]\} \Big] d\gamma;$$

$$T_{DD}(\eta, \rho, \zeta) \equiv -\frac{8}{\pi^2} \int_0^\infty \frac{\cos \gamma\zeta}{\Delta} \gamma^2 G(\gamma, \rho) \left[2\alpha g\left(\frac{\gamma}{\eta}\right) f_2 + g^2\left(\frac{\gamma}{\eta}\right) K f_3 \right] d\gamma. \tag{70 Contd.}$$

In Eqs. (70), γ is the argument of K, f_1, f_2, f_3, f_4, and Δ, which are defined in Eqs. (55) and (61). Note that although the integrals for U_r, T_{rr}, etc., are complicated, the integrands are composed entirely of known functions.

G. Check of the Solution and Results

If the function D is such that all the integrals exist, the stresses and displacements as given by Eqs. (69) satisfy the field equations (7) and (8). Moreover, the boundary conditions (29), (31), and the last two of (30) are met by Eqs. (69) for any function D, again providing all the integrals exist. If the integral equation (65) is numerically solved for $D(\eta)$, the results must satisfy two criteria: the behavior of D as $\eta \to \infty$ must be such that the necessary integrals exist; and the stress τ^R_{zz} at z = 0 calculated from the sixth of Eqs. (69) must satisfy the first of boundary conditions (30).

Accordingly, the numerical calculations are conveniently divided into three main sections: (1) the solution of the integral equation; (2) the check of the boundary conditions on τ^R_{zz}; and (3) the calculation of the desired stresses and displacements in the vicinity of the edge $\rho = 1$ and $\zeta = 0$. The details of these calculations and the accompanying reformulation of some of the expressions are discussed in Sect. II. A summary of the results with some relevant comments will be given here.

It should be pointed out that the solution depends on Poisson's ratio in a complicated fashion; note that the parameter $\alpha \equiv 2(1 - \nu)$ appears in Eqs. (65), (67), (69), and (70) both explicitly and implicitly through the functions Δ, f_1, and f_4, defined in Eqs. (61). Calculations were therefore performed for Poisson's ratios of 1/4 and 1/2, the solution for $\nu = 0$ being known in advance.

1. Solution of the Integral Equation*

The integral equation (65) was solved for $D(\eta)$ at seventy-six values of η in the interval (0,2000). Simpson's rule was used to evaluate

*See Sects. II-A, B, and E for details.

the integral in Eq. (65), using as data points for ξ the same seventy-six values as were used for η. This necessitated the calculation of the kernel $L(\xi,\eta)$ for 76 x 76 combinations of ξ and η. Taking advantage of the symmetry of $L(\xi,\eta)$ and the fact that L is zero if either ξ or η is zero [see Eqs. (67) and (58)], we are still left with 2850 combinations of ξ and η for which L must be numerically evaluated. The calculation of these values, their substitution into Eq. (65), and the subsequent solution of the resultant system of seventy-five simultaneous equations in the seventy-five* unknown values of $D(\eta)$ took 26 min with the IBM-704 for each value of Poisson's ratio.

Figures 2 and 3 are plots of the numerically determined values of $D(\eta)$ and $\eta^4 D(\eta)$, respectively, for ν equal to 1/4 and 1/2. It is apparent that as $\eta \to \infty$, $D(\eta)$ behaves essentially as $c(\nu)\,\eta^{-4}$, with $c(1/4) \approx 0.322$ and $c(1/2) \approx 1.260$. The deviation from this behavior at the largest values of η is probably due to the inaccuracy caused by truncating the integration in Eq. (65) at $\eta = 2000$. The assumption that $D(\eta) \sim c\eta^{-4}$ as $\eta \to \infty$ assures the existence of the integrals for the stresses and displacements. This behavior is also utilized in numerically evaluating the contributions to these integrals of the integrations over large η, the contributions of the "tails" being particularly significant at values of ρ and ζ near ($\rho = 1$, $\zeta = 0$).

2. Check of the Boundary Condition on τ^R_{zz}**

Tables I and II give the results of the numerical check for the stress τ^R_{zz} on the bounding plane $z = 0$. If the function $D(\eta)$ were exactly correct and if the numerical procedures for evaluating τ^R_{zz} were exact, $\tau^R_{zz}(\rho, \theta, 0)$ as calculated from Eqs. (69) and (70) would be $4\nu\rho^{-2}\tau_S \cos 2\theta$ [see the first of Eqs. (30)], or, in other words,

$$\tau^S_{zz}(\rho, \theta, 0) = \tau^P_{zz}(\rho, \theta) + \tau^R_{zz}(\rho, \theta, 0)$$

would be zero [see Eqs. (20) and (28)]. Large deviations of $\tau^S_{zz}(\rho, \theta, 0)$ from zero would have indicated a mistake in the theoretical or numerical solution of the problem. On the other hand, small deviations give an estimate of the errors introduced through the use of approximate numerical procedures. It may be observed from Tables I and II that the deviations tend to increase in magnitude as ρ approaches unity. This is believed to be due to the difficulties involved in evaluating the integrals for τ^R_{zz} for ζ equals zero and ρ near one, rather than due to errors in the values of $D(\eta)$.

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*D(0) is found analytically in Part II, Sect. E, leaving seventy-five values of $D(\eta)$ to be determined numerically.

**See Sects. II-A, C, and F for details.

TABLE I. Check of the Boundary Condition for Poisson's Ratio = 1/4. Values of τ_{zz}^R and τ_{zz}^S on the surface z = 0 for various radial positions.

$\rho = r/a$	$\frac{\tau_{zz}^R(\rho, \theta, 0)}{\tau_S \cos 2\theta}$		$\frac{\tau_{zz}^S(\rho, \theta, 0)}{\tau_S \cos 2\theta}$	
	Numerical Result	Theoretical Value	Numerical Result	Theoretical Value
1	1.00218	1.00000	0.00218	0
1.02	0.95576	0.96117	-0.00541	0
1.1	0.82730	0.82645	0.00085	0
1.2	0.69578	0.69444	0.00134	0
1.4	0.51100	0.51020	0.00080	0
1.6	0.39070	0.39063	0.00007	0
1.8	0.30879	0.30864	0.00015	0
2	0.25011	0.25000	0.00011	0
4	0.06228	0.06250	-0.00022	0
6	0.02706	0.02778	-0.00072	0
8	0.01523	0.01563	-0.00040	0
10	0.00903	0.01000	-0.00097	0

TABLE II. Check of the Boundary Condition for Poisson's Ratio = 1/2. Values of τ_{zz}^R and τ_{zz}^S on the surface z = 0 for various radial positions.

$\rho = r/a$	$\frac{\tau_{zz}^R(\rho, \theta, 0)}{\tau_S \cos 2\theta}$		$\frac{\tau_{zz}^S(\rho, \theta, 0)}{\tau_S \cos 2\theta}$	
	Numerical Result	Theoretical Value	Numerical Result	Theoretical Value
1	2.01167	2.00000	0.01167	0
1.02	1.89914	1.92234	-0.02320	0
1.1	1.65466	1.65289	0.00177	0
1.2	1.39262	1.38889	0.00373	0
1.4	1.02146	1.02041	0.00105	0
1.6	0.78148	0.78125	0.00023	0
1.8	0.61737	0.61728	0.00009	0
2	0.49982	0.50000	-0.00018	0
4	0.12432	0.12500	-0.00068	0
6	0.05381	0.05556	-0.00175	0
8	0.03120	0.03125	-0.00005	0
10	0.02027	0.02000	0.00027	0

In the expression for τ_{zz}^R in Eqs. (69), the term in $D(\eta)\,\eta^4\Omega_2(\eta,\rho)$ $e^{-\eta\zeta}$ goes to zero quite rapidly if $\zeta \neq 0$ because of the $e^{-\eta\zeta}$ factor, and the corresponding integration over η can easily be performed numerically. If ζ equals zero, however, this integrand is an oscillating function with slowly decreasing amplitude. In the numerical integration small steps in η must be taken to follow the oscillations; however, the numerical integration must be carried out to a large value of η before asymptotic expansions become applicable in evaluating the contribution of the "tail" of the integrand.

In the sixth of Eqs. (69) the integration of $D(\eta)\ T_{zz}(\eta,\rho,0)$ over η from zero to infinity was approximated by applying Simpson's rule out to a large value of η. The contribution of the "tail" of this integrand was neglected. It can be shown to be of increasing importance relative to the value of the integral as ρ approaches unity.

To further complicate matters, the integral representation for τ_{zz}^R in Eqs. (69) is discontinuous at the edge ($\rho = 1$, $\zeta = 0$), that is, substitution of the values $\rho = 1$ and $\zeta = 0$ into the integral and numerical integration give a different result than that obtained by taking the limit of the values of the integral as $\rho \to 1$ and $\zeta \to 0$. The difference can be theoretically shown to be exactly $4c(\nu)/\pi$, where $c(\nu)$ is as discussed in Part 1 above. This provides a good check on the values of $c(\nu)$ determined from the graphs of $D(\eta)$.

The integral representation for τ_{zz}^R was evaluated at $\rho = 1$ and $\zeta = 0$ for the two values of Poisson's ratio considered and $4c/\pi$ was added. The resulting values of the stresses are those given in Tables I and II for $\rho = 1$, and are seen to be very close to the theoretical values. Obviously, the integral representations (69) are to be replaced by their limiting values as $\rho \to 1$ and $\zeta \to 0$ when calculating the stresses at the edge. In calculating the stresses near the edge, however, we are faced with the customary difficulties in numerically evaluating the integral representation of a sectionally discontinuous function near a point of discontinuity.

It was mentioned in Sect. D-2 that the problem discussed in this report was initially solved by use of the function Ω_2^0 of Eq. (47) rather than by use of Ω_2 of Eqs. (49). The integral equation obtained in this original solution was solved numerically with no particular difficulty, but the solution [call it $D^0(\eta)$] was found to be such that it could not be replaced by a simple asymptotic representation except for η greater than approximately 1500. Because of this, the difficulties encountered in numerically evaluating the integrals needed for the check of the boundary condition proved to be prohibitive. It may well be that the first formulation of the problem and the corresponding solution $D^0(\eta)$ are quite correct, but without the check of the boundary condition nothing definite can be stated as to their validity.

By inspection of Tables I and II, it appears that the solution to the second formulation, discussed in detail in this report, checks quite nicely. Since the numerical approximations involved in the check of the boundary condition are more inaccurate than those involved in the solution of the integral equation (65) and in the evaluation of the stress and displacement fields at points where $\zeta \neq 0$, it seems likely that the solution $D(\eta)$ and the results presented in Parts 3 and 4 below are more accurate than the results for $\tau_{zz}^{S}(\rho, \theta, 0)$ presented in Tables I and II. In other words, improving the numerical techniques for evaluating $\tau_{zz}^{S}(\rho, \theta, 0)$ should result in less deviation of this quantity from zero, even though the same $D(\eta)$ is used.

3. Stresses and Displacements near the Edge $\rho = 1$, $\zeta = 0$ for the Pure Shear Problem*

Figures 4, 5, and 6 are plots of the stresses τ_{zz}^{S}, $\tau_{\theta\theta}^{S}$, and $\tau_{\theta z}^{S}$ on the hole as a function of axial position; Fig. 7 gives u_z^S on the bounding plane as a function of radial position. From the plane-strain approximation to the pure shear problem, given by Eqs. (27), we have that

$$\tau_{zz}^{P}(1, \theta)/\tau_S \cos 2\theta = -4\nu; \qquad \tau_{\theta\theta}^{P}(1, \theta)/\tau_S \cos 2\theta = -4;$$

$$\tau_{\theta z}^{P}(1, \theta)/\tau_S \sin 2\theta = 0; \qquad 2\mu u_z^{P}(\rho, \theta)/a\tau_S \cos 2\theta = 0. \tag{71}$$

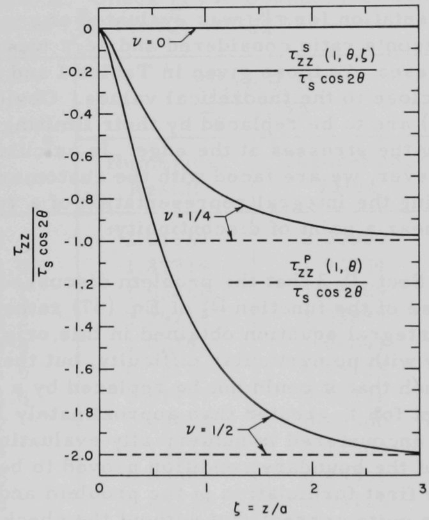

Fig. 4

τ_{zz} on the Hole for the Pure Shear Problem

*See Sects. II-A, D, G, and H for details.

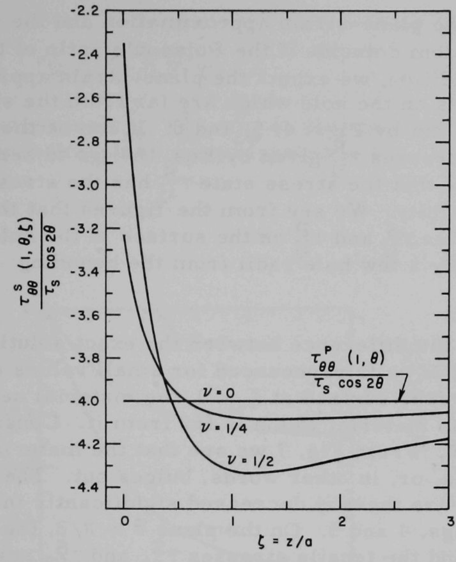

Fig. 5

$\tau_{\theta\theta}$ on the Hole for the Pure Shear Problem

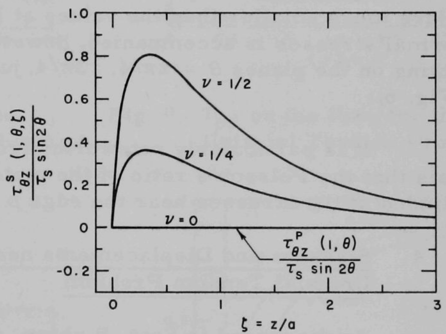

Fig. 6

$\tau_{\theta z}$ on the Hole for the Pure Shear Problem

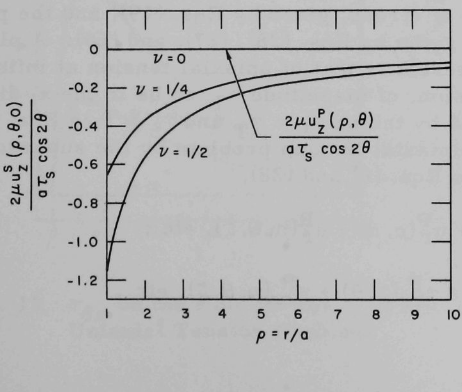

Fig. 7

u_z on the Plane $z = 0$ for the Pure Shear Problem

We also note that the plane-strain approximation and the exact solution to the pure shear problem coincide if the Poisson's ratio of the elastic material is zero. In addition, we expect the plane-strain approximation to be quite good at points on the hole which are far from the surface $\zeta = 0$, and this is indeed borne out by Figs. 4, 5, and 6. It can be theoretically shown that the residual stresses τ_{ij}^{R} given by Eqs. (69) go to zero as ζ approaches infinity and hence that the stress state τ_{ij}^{S} has the stress state τ_{ij}^{P} as its limit as ζ goes to infinity. We see from the figures that the difference between the stress states τ_{ij}^{S} and τ_{ij}^{P} on the surface of the hole becomes small at distances of only a few hole radii from the bounding plane $\zeta = 0$.

On the other hand, the difference between the exact solution and the plane-strain approximation is quite pronounced for small values of ζ. Because of the presence of the free surface at $\zeta = 0$, the material near this surface is less constrained than material which is far from it. Consider for the moment the plane $\theta = 0$. From Fig. 7 we see that the material along $\zeta = 0$ deflects downwards, or, in other words, bulges out. The compressive stresses τ_{zz}^{S} and $\tau_{\theta\theta}^{S}$ are thereby decreased significantly in magnitude, as can be seen from Figs. 4 and 5. On the plane $\theta = \pi/2$, the material on $\zeta = 0$ moves up, and the tensile stresses τ_{zz}^{S} and $\tau_{\theta\theta}^{S}$ near $\zeta = 0$ are much smaller than the values at large ζ. This diminishing of the normal stresses is accompanied, however, by severe shear stresses $\tau_{\theta z}^{S}$ acting on the planes $\theta = \pm\pi/4, \pm 3\pi/4$, just above the free surface (see Fig. 6).

It is particularly noteworthy from the standpoint of photo-elastic analysis that the Poisson's ratio of the material has a large effect on the magnitudes of the stresses near the edge $\rho = 1$, $\zeta = 0$.

4. Stresses and Displacements near the Edge $\rho = 1$, $\zeta = 0$ for the Uniaxial Tension Problem

As discussed in Sect. B above, the solution for any uniform plane state of stress can be obtained by properly superimposing the solutions for the plane hydrostatic state of stress, given by Eqs. (19), and the plane state of pure shearing stress, given by Eqs. (28), (27), and (69). A plane state of particular physical interest is that of uniaxial tension at infinity. If we assume this uniaxial tension, of magnitude τ_T, to be in the x_1 direction, this state will be characterized by taking $\tau_1 = \tau_T$ and $\tau_2 = 0$ in Eqs. (1). Identifying the solution to the uniaxial tension problem by the superscript T, we can then write, referring to Eqs. (6) and (28),

$$u_r^{T}(\rho, \theta, \zeta) = u_r^{H}(\rho, \zeta) + u_r^{P}(\rho, \theta) + u_r^{R}(\rho, \theta, \zeta), \text{ etc.};$$

$$\tau_{rr}^{T}(\rho, \theta, \zeta) = \tau_{rr}^{H}(\rho, \zeta) + \tau_{rr}^{P}(\rho, \theta) + \tau_{rr}^{R}(\rho, \theta, \zeta), \text{ etc.,} \qquad (72)$$

with

$$\tau_H = \tau_S = \tfrac{1}{2}\tau_T.$$

The stresses τ_{zz}^T, $\tau_{\theta\theta}^T$, $\tau_{\theta z}^T$ on the hole and the normal displacement u_z^T on the bounding plane are presented in Figs. 8, 9, 10, and 11.

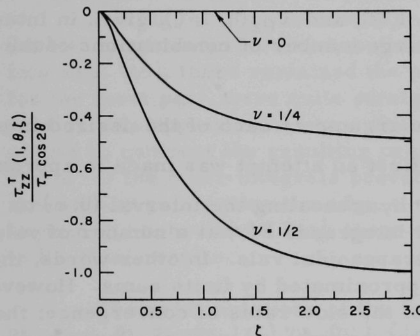

Fig. 8. τ_{zz} on the Hole for the Uniaxial Tension Problem

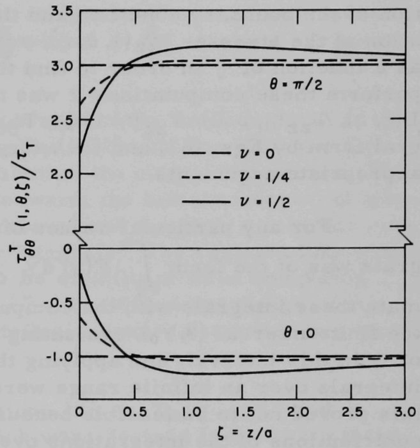

Fig. 9. $\tau_{\theta\theta}$ on the Hole for the Uniaxial Tension Problem

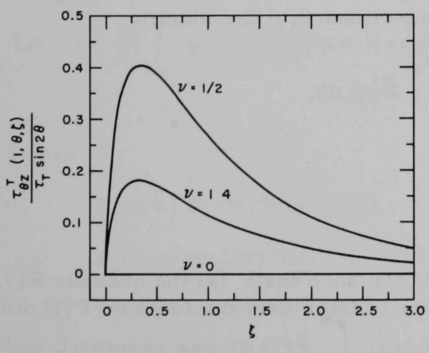

Fig. 10. $\tau_{\theta z}$ on the Hole for the Uniaxial Tension Problem

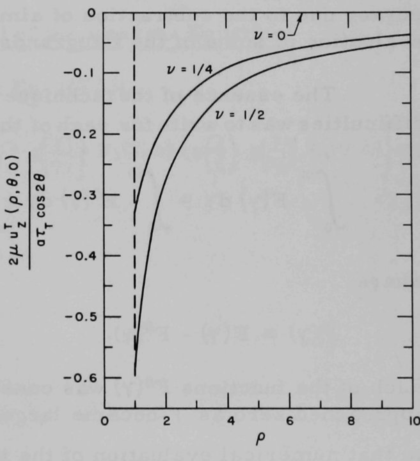

Fig. 11. u_z on the Plane z = 0 for the Uniaxial Tension Problem

II. NUMERICAL ANALYSIS

A. Introduction

The numerical computations performed were, briefly, the solution of the integral Eqs. (65) for the function $D(\eta)$; the evaluation of the stress $\tau^R_{zz}(\rho, \theta, 0)$ from Eqs. (69) as a function of ρ in order to check the satisfaction of the boundary condition and thereby the entire solution; and the evaluation of the stresses $\tau^R_{zz}(1, \theta, \zeta)$, $\tau^R_{\theta\theta}(1, \theta, \zeta)$, and $\tau^R_{\theta z}(1, \theta, \zeta)$ from Eqs. (69) as a function of ζ in order to find the stress distribution on the hole. To perform these computations it was necessary to evaluate the functions $L(\xi, \eta)$, $T_{zz}(\eta, \rho, 0)$, $T_{zz}(\eta, 1, \zeta)$, $T_{\theta\theta}(\eta, 1, \zeta)$, and $T_{\theta z}(\eta, 1, \zeta)$, given in integral form by Eqs. (67) and (70), for a large number of combinations of the appropriate arguments.

For any particular values of the arguments each of the desired integrals was of the form $\int_0^\infty F(\gamma)\, d\gamma$. At first an attempt was made to approximate these integrals with the computer by truncating the interval $(0, \infty)$ to the finite interval $(0, \gamma_0)$, evaluating the integrands $F(\gamma)$ at a number of values of γ in this interval, and applying the trapezoidal rule. In other words, the integrals over an infinite range were approximated by finite sums. However, this proved not to be feasible because of the slow rates of convergence; the contributions of the integrations over the range (γ_0, ∞) were too large to be ignored unless very large values of γ_0 were selected. These large values of γ_0 would have required the consumption of an excessive amount of machine time to evaluate the integrals for all the required combinations of the parameters. Moreover, an additional complication lay in significant losses in accuracy due to the subtraction of almost equal quantities encountered in the evaluation of some of the integrands.

The essence of the technique employed to overcome these integration difficulties was to write for each of the functions $F(\gamma)$ the identity

$$\int_0^\infty F(\gamma)\, d\gamma = \int_0^\infty F^0(\gamma)\, d\gamma + \int_0^\infty \tilde{F}(\gamma)\, d\gamma,$$

where

$$\tilde{F}(\gamma) \equiv F(\gamma) - F^0(\gamma).$$

Each of the functions $F^0(\gamma)$ was constructed such that: (a) the quantity $\tilde{F}(\gamma)$ approached zero as γ became large much faster than the function $F(\gamma)$ did, so that numerical evaluation of the integral $\int_0^\infty \tilde{F}(\gamma)\, d\gamma$ was practical; and

(b) the integral $\int_0^\infty F^0(\gamma)\, d\gamma$ was integrable in closed form. An insight into the cause of some of the numerical difficulties was obtained when it was found that some of the integrals $\int_0^\infty F^0(\gamma)\, d\gamma$ were expressible in terms of the sine integral, cosine integral, and exponential integrals. These functions are very difficult to compute from their integral definitions, and, indeed, roundabout techniques are invariably employed in the standard computer subroutines for their evaluation.

Having in this way successfully evaluated the inner integral of the double integrals formed by substituting Eqs. (67) into Eqs. (65) and Eqs. (70) into Eqs. (69), there remained the problem of the outer integrations. These, for the most part, were quite straightforward; the infinite interval of integration was approximated by a finite interval and Simpson's rule was employed to compute the resulting proper integrals. The contributions of the "tails" of the outer integrals proved to be significant when computing stresses at points near the edge $\rho = 1$, $\zeta = 0$. With a knowledge of the asymptotic behavior of $D(\eta)$, these contributions were estimated and included in the final results.

The reformulations of the functions $L(\xi, \eta)$, $T_{zz}(\eta, \rho, 0)$, and $T_{zz}(\eta, 1, \zeta)$, $T_{\theta\theta}(\eta, 1, \zeta)$ $T_{\theta z}(\eta, 1, \zeta)$ which were devised to facilitate their numerical evaluation are discussed in Sects. B, C, and D, respectively. These are followed by discussions of: the solution of the integral equation, in E; the check of the solution, in F; the calculation of $\tau_{zz}(1, \theta, \zeta)$, $\tau_{\theta\theta}(1, \theta, \zeta)$, and $\tau_{\theta z}(1, \theta, \zeta)$, in G; and the calculation of $u_z(\rho, \theta, 0)$, in H.

B. Reformulation of the Kernel $L(\xi, \eta)$ of the Integral Equation

The kernel $L(\xi, \eta)$ as defined in Eq. (67) is

$$L(\xi, \eta) \equiv \frac{32}{\pi^3} \int_0^\infty g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right) \left\{F_1(\gamma, \alpha) + \left[g\left(\frac{\gamma}{\xi}\right) + g\left(\frac{\gamma}{\eta}\right)\right] F_2(\gamma, \alpha) + g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right) F_3(\gamma, \alpha)\right\} d\gamma, \tag{73}$$

where

$$\begin{aligned} F_1(\gamma, \alpha) &\equiv -\alpha\gamma^2 f_1(\gamma)/\Delta(\gamma); \\ F_2(\gamma, \alpha) &\equiv -2\alpha\gamma^2 K(\gamma)\, f_2(\gamma)/\Delta(\gamma); \\ F_3(\gamma, \alpha) &\equiv -\gamma^2 K^2(\gamma)\, f_3(\gamma)/\Delta(\gamma). \end{aligned} \tag{74}$$

The function g is defined by Eqs. (58) and K by Eqs. (55), while f_1, f_2, f_3, and Δ are defined in Eqs. (61). Figures 12 and 13 are graphs of K and Δ. The parameter α, defined in terms of Poisson's ratio by Eqs. (12), is included as an argument of the functions F_1, F_2 and F_3 for notational convenience in the discussion that follows. Recall that $f_1(\gamma)$ and $\Delta(\gamma)$ by definition depend on α as a parameter.

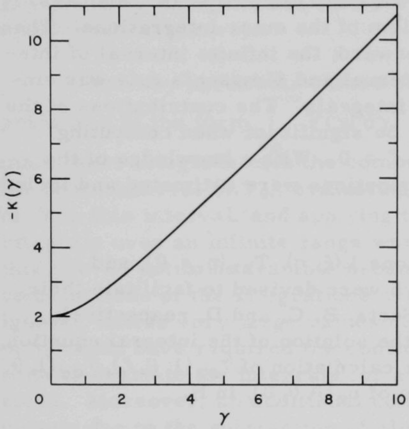

Fig. 12. The Function $K(\gamma)$

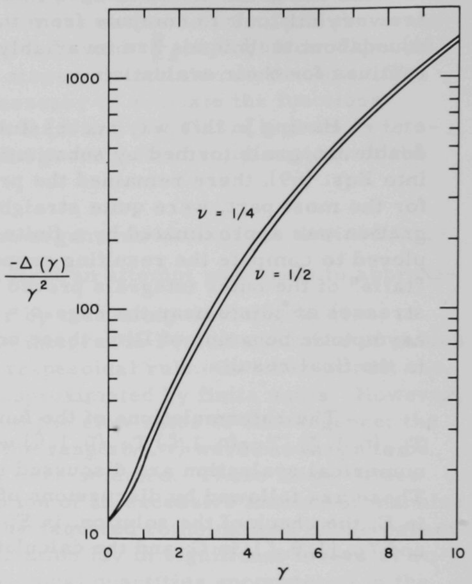

Fig. 13. The Function $\Delta(\gamma)/\gamma^2$ for Poisson's Ratios of 1/4 and 1/2.

Define

$$F_1^0(\gamma, \alpha) \equiv -\alpha; \qquad F_2^0(\gamma, \alpha) \equiv -2\alpha\gamma + 2\alpha^2;$$

$$F_3^0(\gamma, \alpha) \equiv \gamma^3 + (1-\alpha)\gamma^2 + \tfrac{1}{8}(8\alpha^2 - 16\alpha + 47)\gamma + \tfrac{1}{4}(-4\alpha^3 + 12\alpha^2 - 3\alpha - 15), \tag{75}$$

and

$$\widetilde{F}_j(\gamma, \alpha) \equiv F_j(\gamma, \alpha) - F_j^0(\gamma, \alpha), \qquad j = 1, 2, 3. \tag{76}$$

The F_j^0 are identical to the terms in nonnegative powers of γ which occur in the asymptotic expansions* of the functions F_j. Consequently, asymptotic expansions of the functions $\widetilde{F}_j$ are given by

*See Appendix B, Sect. 8.

$$\tilde{F}_j(\gamma, \alpha) \sim \sum_{m=1}^{N_j} \frac{b_{jm}(\alpha)}{\gamma^m}, \qquad j = 1, 2, 3. \tag{77}$$

Numerical values for the coefficients b_{jm} for Poisson's ratios of 1/2 and 1/4 are given in Appendix B, Sect. 9. Figures 14-17 are graphs of $F_j(\gamma, 1)$,

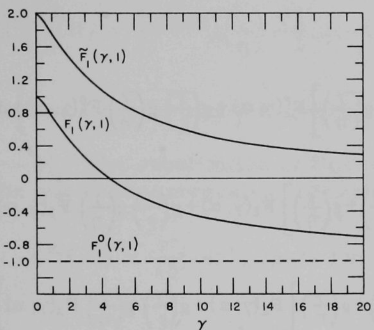

Fig. 14. The Functions F_1, F_1^0, and $\tilde{F}_1$ for $\nu = 1/2$

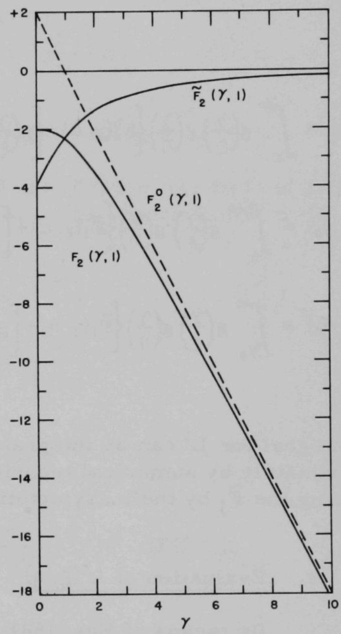

Fig. 15. The Functions F_2, F_2^0, and $\tilde{F}_2$ for $\nu = 1/2$

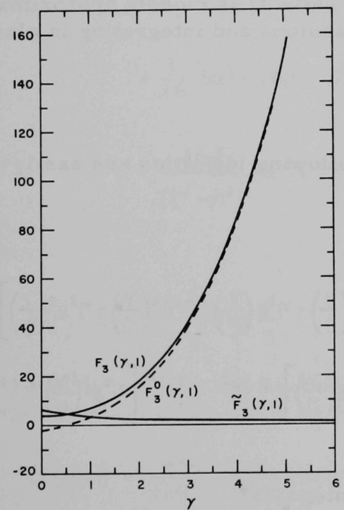

Fig. 16. The Functions F_3, F_3^0, and $\tilde{F}_3$ for $\nu = 1/2$

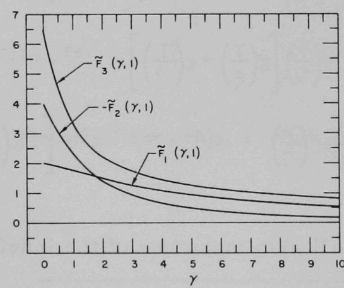

Fig. 17. The Functions $\tilde{F}_1$, $\tilde{F}_2$, and $\tilde{F}_3$ for $\nu = 1/2$

$F_j^0(\gamma, 1)$, and $\widetilde{F}_j(\gamma, 1)$, $j = 1, 2, 3$. It is apparent from these figures that the integrals in Eqs. (73) should be much easier to evaluate numerically if the F_j functions are replaced by the $\widetilde{F}_j$ functions.

By means of Eqs. (73) and (76) we can write

$$L(\xi, \eta) = \frac{32}{\pi^3}[L^0(\xi, \eta) + \widetilde{L}(\xi, \eta, \gamma_0) + \widetilde{\widetilde{L}}(\xi, \eta, \gamma_0)], \tag{78}$$

with

$$L^0(\xi, \eta) \equiv \int_0^\infty g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right)\left\{F_1^0(\gamma, \alpha) + \left[g\left(\frac{\gamma}{\xi}\right) + g\left(\frac{\gamma}{\eta}\right)\right] F_2^0(\gamma, \alpha) + g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right) F_3^0(\gamma, \alpha)\right\} d\gamma;$$

$$\widetilde{L}(\xi, \eta, \gamma_0) \equiv \int_0^{\gamma_0} g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right)\left\{\widetilde{F}_1(\gamma, \alpha) + \left[g\left(\frac{\gamma}{\xi}\right) + g\left(\frac{\gamma}{\eta}\right)\right] \widetilde{F}_2(\gamma, \alpha) + g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right) \widetilde{F}_3(\gamma, \alpha)\right\} d\gamma;$$

$$\widetilde{\widetilde{L}}(\xi, \eta, \gamma_0) \equiv \int_{\gamma_0}^\infty g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right)\left\{\widetilde{F}_1(\gamma, \alpha) + \left[g\left(\frac{\gamma}{\xi}\right) + g\left(\frac{\gamma}{\eta}\right)\right] \widetilde{F}_2(\gamma, \alpha) + g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right) \widetilde{F}_3(\gamma, \alpha)\right\} d\gamma. \tag{79}$$

The integral for L^0 can be integrated exactly in closed form, $\widetilde{L}$ is evaluated approximately by numerical techniques, while $\widetilde{\widetilde{L}}$ is closely approximated by replacing the $\widetilde{F}_j$ by their asymptotic expansions and integrating in closed form.

1. Evaluation of $L^0(\xi, \eta)$

By means of Eqs. (58), the following identities are easily verified:

$$g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right) = (\xi^2 - \eta^2)^{-1}\left[\xi^2 g\left(\frac{\gamma}{\eta}\right) - \eta^2 g\left(\frac{\gamma}{\xi}\right)\right];$$

$$g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right)\left[g\left(\frac{\gamma}{\xi}\right) + g\left(\frac{\gamma}{\eta}\right)\right] = (\xi^2 - \eta^2)^{-1}\left[\xi^2 g\left(\frac{\gamma}{\eta}\right) - \eta^2 g\left(\frac{\gamma}{\xi}\right) + \xi^2 g^2\left(\frac{\gamma}{\eta}\right) - \eta^2 g^2\left(\frac{\gamma}{\xi}\right)\right];$$

$$g^2\left(\frac{\gamma}{\xi}\right) g^2\left(\frac{\gamma}{\eta}\right) = 2(\xi^2 - \eta^2)^{-3}\, \xi^2\eta^2\left[\eta^2 g\left(\frac{\gamma}{\xi}\right) - \xi^2 g\left(\frac{\gamma}{\eta}\right)\right] + (\xi^2 - \eta^2)^{-2}\left[\eta^4 g^2\left(\frac{\gamma}{\xi}\right) + \xi^4 g^2\left(\frac{\gamma}{\eta}\right)\right]. \tag{80}$$

We will also make use of the indefinite integrals*

*Empty sums are taken to be zero.

$$\int \gamma^{2m} g\left(\frac{\gamma}{\eta}\right) d\gamma = (-1)^m \eta^{2m+1}\left[\arctan\frac{\gamma}{\eta} + \sum_{j=1}^{m} \frac{(-1)^j}{2j-1}\left(\frac{\gamma}{\eta}\right)^{2j-1}\right];$$

$$\int \gamma^{2m+1} g\left(\frac{\gamma}{\eta}\right) d\gamma = \frac{1}{2}(-1)^m \eta^{2m+2}\left[\log(\gamma^2+\eta^2) + \sum_{j=1}^{m} \frac{(-1)^j}{j}\left(\frac{\gamma}{\eta}\right)^{2j}\right];$$

$$\int \gamma^{m} g^2\left(\frac{\gamma}{\eta}\right) d\gamma = \frac{1}{2}\gamma^{m+1} g\left(\frac{\gamma}{\eta}\right) - \frac{1}{2}(m-1)\int \gamma^m g\left(\frac{\gamma}{\eta}\right) d\gamma, \quad m = 0,1,2,\ldots, \tag{81}$$

which can be checked by differentiation with respect to γ.

The substitution of Eqs. (80) and (75) into the first of Eqs. (79) and the use of the integrals (81) evaluated at the proper limits yield

$$\begin{aligned} L^0(\xi,\eta) &= \frac{\pi}{16}(-4\alpha^3 + 36\alpha^2 - 11\alpha - 15)\frac{\xi\eta}{\xi+\eta} \\ &+ \frac{\pi}{16}(-4\alpha^3 + 12\alpha^2 - 3\alpha - 15)\frac{\xi^2\eta^2}{(\xi+\eta)^3} \\ &+ \frac{\pi}{4}(1-\alpha)\frac{\xi^3\eta^3}{(\xi+\eta)^3} - 2\alpha F(\xi,\eta) \\ &+ \frac{1}{16}(8\alpha^2 - 16\alpha + 47)\frac{\xi^2\eta^2}{(\xi^2-\eta^2)^2}[\xi^2+\eta^2 - 4F(\xi,\eta)] \\ &+ \frac{\xi^2\eta^2}{(\xi^2-\eta^2)^2}[(\xi^2+\eta^2)F(\xi,\eta) - \xi^2\eta^2], \qquad \xi \neq \eta, \end{aligned} \tag{82}$$

where

$$F(\xi,\eta) \equiv \frac{\xi^2\eta^2}{\xi^2-\eta^2}\log\frac{\xi}{\eta}. \tag{83}$$

Either letting $\xi = \eta$ in the first of Eqs. (79) and integrating, or applying L'Hospital's Rule to Eq. (82), we have

$$\begin{aligned} L^0(\eta,\eta) &= \frac{\pi}{128}(-20\alpha^3 + 156\alpha^2 - 47\alpha - 75)\eta \\ &+ \frac{1}{48}(8\alpha^2 - 64\alpha + 47)\eta^2 + \frac{\pi}{32}(1-\alpha)\eta^3 + \frac{1}{12}\eta^4. \end{aligned} \tag{84}$$

The quantity $L^0(\xi, \eta)$ is much larger than $\tilde{L}(\xi, \eta, \gamma_0) + \tilde{\tilde{L}}(\xi, \eta, \gamma_0)$ except for small values of either ξ or η.

2. Evaluation of $\tilde{L}(\xi, \eta, \gamma_0)$

Two techniques were developed for numerically approximating the integral in the second of Eqs. (79), the first being suitable for desk calculation and the second for use with the IBM-704. Results obtained for some combinations of ξ and η by the first technique were used to check the validity of the computer program. The dividing point γ_0 between the ranges of integration of $\tilde{L}$ and $\tilde{\tilde{L}}$ was taken equal to 10, since it was found that the asymptotic expansions used for the $\tilde{F}_j$ in the evaluation of $\tilde{\tilde{L}}$ were sufficiently accurate for $\gamma \geq 10$.

The first technique consisted essentially in dividing the interval (0, 10) into the subintervals (0, 1), (1, 2), (2, 4), (4, 6), and (6, 10), and fitting polynomial approximations to each function $\tilde{F}_j(\gamma, \alpha)$ in each subinterval. In order to fit the polynomials, the functions $\tilde{F}_j$ were calculated at a number of points in each subinterval by use of Eqs. (76), (75), (74), (55), (61), and tabulated values of the modified Bessel function.[17] A fourth-order polynomial was fitted to each $\tilde{F}_1$, $\tilde{F}_2$, and $\tilde{F}_3$ in the interval (0, 1). However, from plots on log-log paper it was obvious that much better fits would be obtained for the other subintervals by fitting the fourth-order polynomials to $\gamma^{-m}\tilde{F}_j$ where m is an integer which depends on the particular subinterval and function being considered. The best value of m usually turned out to be the closest integer to the slope on log-log paper of the straight line connecting the endpoints of the function for the subinterval. Substitution of these approximations in place of the corresponding $\tilde{F}_j$ in the second of Eqs. (79) yielded integrals which could be evaluated in closed form for each subinterval in terms of explicit functions of ξ and η. The only special functions which resulted were the logarithm and arc tangent, both of which are well tabulated.[18,19]

The second technique was to divide the interval (0, 10) into an even number of equal subintervals and to evaluate the integrand at each mesh point for the particular values of ξ, η, and α being considered. Simpson's rule was applied to these values of the integrand, that is, it was assumed that a parabola would be a good fit to each set of three consecutive points. In the computer evaluation of $\tilde{L}(\xi, \eta)$, subintervals of length 0.1 gave adequate accuracy; the FORTRAN library subroutine was used to compute the modified Bessel functions needed.

In comparison, the first technique is much superior to the second for hand computations with a desk calculator. Computing the values of the $\tilde{F}_j$ and fitting the fourth-order polynomials to the $\gamma^{-m}\tilde{F}_j$ take a lot of time, but this is only done once. Since the integration is done analytically, the only numerical work remaining is to evaluate the resulting expression for each combination of ξ and η desired, taking advantage of the fact that $\tilde{L}(\xi, \eta) = \tilde{L}(\eta, \xi)$. Note that the functions $g(\gamma/\xi)$ and $g(\gamma/\eta)$ appearing in Eqs. (79) are not approximated. Since ξ and η enter the integrand only through these functions, the accuracy of the first technique should be essentially independent of ξ and η. The only source of inaccuracy is in the polynomial approximations to the $\tilde{F}_j$; these approximations can be improved as much as desired by decreasing the lengths of the subintervals and/or increasing the number of terms in the polynomials. By means of the second technique, however, although the values of the $\tilde{F}_j(\gamma, \alpha)$ can be computed once and for all at each mesh point value of γ, the functions $g(\gamma/\xi)$ and $g(\gamma/\eta)$ at each mesh point must be computed for each combination of ξ and η, and a separate numerical integration must be performed each time. Moreover, the accuracy of the parabolic fits implied by the use of Simpson's rule will vary with ξ and η. (In the first technique the polynomial fits are to the $\tilde{F}_j$ functions alone; in the second, the parabolas are fitted to the entire integrand.)

The reason for use of the second rather than the first technique for the digital computer work was simply ease of programming. It was believed that any disadvantage in inaccuracy could be overcome by increasing the number of subintervals. Incorporating the flexibility of a variable number of subintervals into the program for the second technique necessitates a negligible amount of additional effort. It was feared that, for the small length of subinterval required for reasonable accuracy, the computing time necessary to evaluate numerically the integral for the 2850 combination of ξ and η might prove to be exorbitant, and hence necessitate the programming of the first technique. This did not prove to be the case, however.

3. Evaluation of $\tilde{\tilde{L}}(\xi, \eta, \gamma_0)$

For $\gamma \geq \gamma_0 = 10$, the asymptotic expansions for $\tilde{F}_1$, $\tilde{F}_2$, and $\tilde{F}_3$, indicated by Eqs. (77) and discussed in Appendix B, Sect. 9, are sufficiently accurate to warrant their use in the evaluation of $\tilde{\tilde{L}}$. Substituting Eqs. (77) into the third of Eqs. (79), we have

$$\begin{aligned}\tilde{\tilde{L}}(\xi, \eta, 10) \approx \sum_{m=1}^{N_1} b_{1m}(\alpha) \int_{10}^{\infty} \gamma^{-m} g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right) d\gamma \\ + \sum_{m=1}^{N_2} b_{2m}(\alpha) \int_{10}^{\infty} \gamma^{-m} g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right)\left[g\left(\frac{\gamma}{\xi}\right) + g\left(\frac{\gamma}{\eta}\right)\right] d\gamma \\ + \sum_{m=1}^{N_3} b_{3m}(\alpha) \int_{10}^{\infty} \gamma^{-m} g^2\left(\frac{\gamma}{\xi}\right) g^2\left(\frac{\gamma}{\eta}\right) d\gamma.\end{aligned} \tag{85}$$

We make use of the integrals

$$\int_{\gamma_0}^{\infty} \gamma^{-m} g\left(\frac{\gamma}{\eta}\right) d\gamma = \frac{\eta^2}{\gamma_0^{m+1}} H_m\left(\frac{\eta}{\gamma_0}\right);$$

$$\int_{\gamma_0}^{\infty} \gamma^{-m} g^2\left(\frac{\gamma}{\eta}\right) d\gamma = \frac{\eta^2}{2\gamma_0^{m+1}} \left[(m+1) H_m\left(\frac{\eta}{\gamma_0}\right) - g\left(\frac{\eta}{\gamma_0}\right)\right],$$

$$m = 1, 2, 3, \ldots, \tag{86}$$

where

$$H_{2j+1}(X) \equiv \frac{(-1)^j}{X^{2j+2}}\left[\frac{1}{2}\log\,(1+X^2) + \sum_{i=1}^{j} \frac{(-1)^i X^{2i}}{2i}\right], \qquad j = 0, 1, 2, \ldots;$$

$$H_{2j}(X) \equiv \frac{(-1)^j}{X^{2j+1}}\left[\arctan X + \sum_{i=1}^{j} \frac{(-1)^i X^{2i-1}}{2i-1}\right], \qquad j = 1, 2, 3, \ldots . \tag{87}$$

Using the Taylor series expansions for the logarithm and arc tangent* we have

$$H_m(X) = \sum_{i=0}^{\infty} \frac{(-1)^i X^{2i}}{2i+m+1}, \qquad |X| < 1, \qquad m = 1, 2, 3, \ldots . \tag{88}$$

The functions $H_m(X)$ satisfy the differentiation formula

$$XH_m'(X) = -(m+1) H_m(X) + g(X) \tag{89}$$

and the recurrence relation

$$X^2 H_{m+2}(X) = \frac{1}{m+1} - H_m(X), \qquad m = 1, 2, 3, \ldots . \tag{90}$$

The relations (89) and (90) can be checked by use of the definitions (87) and (58); the integrals (86) can be verified by differentiating with respect to γ_0, using Eqs. (89), and noting that

$$g(X^{-1}) = X^2 g(X). \tag{91}$$

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*See Ref. 20, pp. 91 and 92.

By Eqs. (80) and (86) we have, for $\xi \neq \eta$,

$$\int_{\gamma_0}^{\infty} \gamma^{-m} g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right) d\gamma = \frac{1}{\gamma_0^{m+1}} \left(\frac{\xi^2\eta^2}{\xi^2 - \eta^2}\right)\left[-H_m\left(\frac{\xi}{\gamma_0}\right) + H_m\left(\frac{\eta}{\gamma_0}\right)\right];$$

$$\int_{\gamma_0}^{\infty} \gamma^{-m} g\left(\frac{\gamma}{\xi}\right) g\left(\frac{\gamma}{\eta}\right)\left[g\left(\frac{\gamma}{\xi}\right) + g\left(\frac{\gamma}{\eta}\right)\right] d\gamma = \frac{1}{2\gamma_0^{m+1}} \left(\frac{\xi^2\eta^2}{\xi^2 - \eta^2}\right)\left\{-\left[(m+3) H_m\left(\frac{\xi}{\gamma_0}\right) - g\left(\frac{\xi}{\gamma_0}\right)\right]\right.$$

$$\left. + \left[(m+3) H_m\left(\frac{\eta}{\gamma_0}\right) - g\left(\frac{\eta}{\gamma_0}\right)\right]\right\};$$

$$\int_{\gamma_0}^{\infty} \gamma^{-m} g^2\left(\frac{\gamma}{\xi}\right) g^2\left(\frac{\gamma}{\eta}\right) d\gamma = \frac{2}{\gamma_0^{m+1}} \frac{\xi^4\eta^4}{(\xi^2 - \eta^2)^3}\left[H_m\left(\frac{\xi}{\gamma_0}\right) - H_m\left(\frac{\eta}{\gamma_0}\right)\right]$$

$$+ \frac{1}{2\gamma_0^{m+1}} \frac{\xi^4\eta^4}{(\xi^2 - \eta^2)^2}\left\{\frac{1}{\xi^2}\left[(m+1) H_m\left(\frac{\xi}{\gamma_0}\right) - g\left(\frac{\xi}{\gamma_0}\right)\right] + \frac{1}{\eta^2}\left[(m+1) H_m\left(\frac{\eta}{\gamma_0}\right) - g\left(\frac{\eta}{\gamma_0}\right)\right]\right\},$$

$$m = 1, 2, 3, \ldots . \qquad (92)$$

By letting $\xi = \eta$ in the left side of Eqs. (92) and integrating, or by applying L'Hospital's rule to the right side of Eqs. (92), we arrive at the integrals corresponding to Eqs. (92) for $\xi = \eta$:

$$\int_{\gamma_0}^{\infty} \gamma^{-m} g^2\left(\frac{\gamma}{\eta}\right) d\gamma = \frac{1}{2\gamma_0^{m-1}}\left[(m+1)\frac{\eta^2}{\gamma_0^2} H_m\left(\frac{\eta}{\gamma_0}\right) - g\left(\frac{\gamma_0}{\eta}\right)\right];$$

$$\int_{\gamma_0}^{\infty} 2\gamma^{-m} g^3\left(\frac{\gamma}{\eta}\right) d\gamma = \frac{1}{4\gamma_0^{m-1}}\left[(m+1)(m+3)\frac{\eta^2}{\gamma_0^2} H_m\left(\frac{\eta}{\gamma_0}\right) - (m+3)\, g\left(\frac{\gamma_0}{\eta}\right) - 2g^2\left(\frac{\gamma_0}{\eta}\right)\right];$$

$$\int_{\gamma_0}^{\infty} \gamma^{-m} g^4\left(\frac{\gamma}{\eta}\right) d\gamma = \frac{1}{48\gamma_0^{m-1}}\left[(m+1)(m+3)(m+5)\frac{\eta^2}{\gamma_0^2} H_m\left(\frac{\eta}{\gamma_0}\right)\right.$$

$$\left. - (m+3)(m+5)\, g\left(\frac{\gamma_0}{\eta}\right) - 2(m+5)\, g^2\left(\frac{\gamma_0}{\eta}\right) - 8g^3\left(\frac{\gamma_0}{\eta}\right)\right],$$

$$m = 1, 2, 3, \ldots . \qquad (93)$$

Substitution of Eqs. (92), (93), and (77) into (85) yields

$$\tilde{\tilde{L}}(\xi, \eta, \gamma_0) \approx \frac{\xi^2\eta^2}{\xi^2 - \eta^2}\left\{\sum_{m=1}^{N_1} \frac{b_{1m}(\alpha)}{\gamma_0^{m+1}}\left[-H_m\left(\frac{\xi}{\gamma_0}\right) + H_m\left(\frac{\eta}{\gamma_0}\right)\right]\right.$$

$$+ \frac{1}{2\gamma_0}\tilde{F}_2(\gamma_0, \alpha)\left[g\left(\frac{\xi}{\gamma_0}\right) - g\left(\frac{\eta}{\gamma_0}\right)\right]$$

$$\left. + \sum_{m=1}^{N_2} \frac{(m+3)\, b_{2m}(\alpha)}{2\gamma_0^{m+1}}\left[-H_m\left(\frac{\xi}{\gamma_0}\right) + H_m\left(\frac{\eta}{\gamma_0}\right)\right]\right\}$$

$$+ \frac{\xi^4\eta^4}{(\xi^2 - \eta^2)^3}\sum_{m=1}^{N_3} \frac{2b_{3m}(\alpha)}{\gamma_0^{m+1}}\left[H_m\left(\frac{\xi}{\gamma_0}\right) - H_m\left(\frac{\eta}{\gamma_0}\right)\right]$$

$$+ \frac{\xi^2\eta^2}{(\xi^2 - \eta^2)^2}\left\{-\frac{1}{2\gamma_0}\tilde{F}_3(\gamma_0, \alpha)\left[\eta^2 g\left(\frac{\xi}{\gamma_0}\right) + \xi^2 g\left(\frac{\eta}{\gamma_0}\right)\right]\right.$$

$$\left. + \sum_{m=1}^{N_3} \frac{(m+1)\, b_{3m}(\alpha)}{2\gamma_0^{m+1}}\left[\eta^2 H_m\left(\frac{\xi}{\gamma_0}\right) + \xi^2 H_m\left(\frac{\eta}{\gamma_0}\right)\right]\right\}, \qquad \xi \neq \eta \tag{94}$$

and

$$\tilde{\tilde{L}}(\eta, \eta, \gamma_0) \approx -\frac{\gamma_0}{2}\tilde{F}_1(\gamma_0, \alpha)\, g\left(\frac{\gamma_0}{\eta}\right) + \eta^2 \sum_{m=1}^{N_1} \frac{(m+1)\, b_{1m}(\alpha)}{2\gamma_0^{m+1}} H_m\left(\frac{\eta}{\gamma_0}\right)$$

$$- \frac{\gamma_0}{2}\tilde{F}_2(\gamma_0, \alpha)\, g^2\left(\frac{\gamma_0}{\eta}\right) - g\left(\frac{\gamma_0}{\eta}\right)\sum_{m=1}^{N_2} \frac{(m+3)\, b_{2m}(\alpha)}{4\gamma_0^{m-1}}$$

$$+ \eta^2 \sum_{m=1}^{N_2} \frac{(m+1)(m+3)\, b_{2m}(\alpha)}{4\gamma_0^{m+1}} H_m\left(\frac{\eta}{\gamma_0}\right)$$

$$- \frac{\gamma_0}{6}\tilde{F}_3(\gamma_0, \alpha)\, g^3\left(\frac{\gamma_0}{\eta}\right) - g^2\left(\frac{\gamma_0}{\eta}\right)\sum_{m=1}^{N_3} \frac{(m+5)\, b_{3m}(\alpha)}{24\gamma_0^{m-1}}$$

$$- g\left(\frac{\gamma_0}{\eta}\right)\sum_{m=1}^{N_3} \frac{(m+3)(m+5)\, b_{3m}(\alpha)}{48\gamma_0^{m-1}}$$

$$+ \eta^2 \sum_{m=1}^{N_3} \frac{(m+1)(m+3)(m+5)\, b_{3m}(\alpha)}{48\gamma_0^{m+1}} H_m\left(\frac{\eta}{\gamma_0}\right). \tag{95}$$

In the program for the digital computer, the values of H_1 and H_2 were calculated from Eqs. (87), and the recurrence relation (90) employed for H_3, H_4, and so on. For arguments less than one, however, this procedure would result in subtraction of almost equal numbers,* the accuracy in the value of H_m decreasing with increasing m. Therefore, for arguments in this range, the power series (88) for H_m was used instead of Eqs. (87); in the computer program, H_m was calculated from Eqs. (88) for the largest odd and even values of m, and then the recurrence relation (90) was used to find the H_m for lower values of the order m down to H_1 and H_2.**

Figures 18, 19, and 20 are plots of $L(\xi, \eta)$, found from (78), (82), (84), (79), (94), and (95), for Poisson's ratios of 1/4 and 1/2.

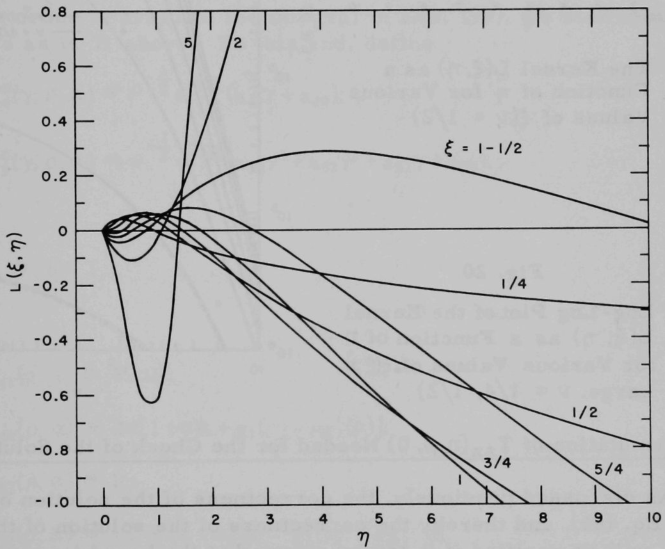

Fig. 18. The Kernel $L(\xi, \eta)$ as a Function of η for Various Values of $\xi(\nu = 1/4)$

*The terms from the finite sums in Eqs. (87) exactly cancel the leading terms in the power series expansions for small x of the associated logarithm and arc tangent.

**Note that proceeding in the other direction, i.e., finding H_1 and H_2 from the series (88) and using Eqs. (90) for higher values of m, would result in great loss in accuracy due to subtraction of almost equal numbers.

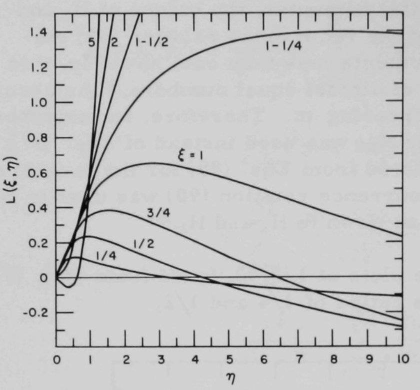

Fig. 19. The Kernel $L(\xi, \eta)$ as a Function of η for Various Values of $\xi(\nu = 1/2)$

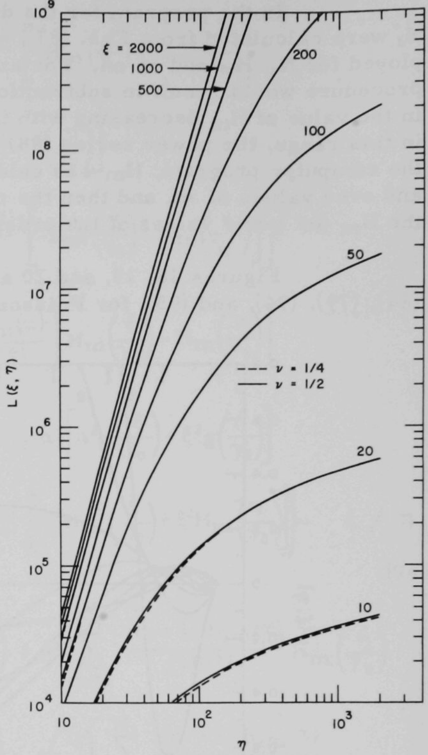

Fig. 20

Log-Log Plot of the Kernel $L(\xi, \eta)$ as a Function of η for Various Values of $\xi(\xi, \eta$ Large, $\nu = 1/4, 1/2)$

C. Reformulation of $T_{zz}(\eta, \rho, 0)$ Needed for the Check of the Solution

As discussed previously, the correctness of the solution of the integral Eq. (65), and thereby the correctness of the solution of the entire problem, will be verified if it can be shown that the boundary condition

$$\tau_{zz}^{R}(\rho, \theta, \zeta) = 2(2-\alpha)\,\tau_S\rho^{-2}\cos 2\theta \text{ on } \zeta = 0 \tag{96}$$

is satisfied. From Eqs. (69) we have that

$$\frac{\tau_{zz}^{R}(\rho, \theta, 0)}{\tau_S \cos 2\theta} = \int_0^\infty D(\eta)[\eta^4\Omega_2(\eta, \rho) + T_{zz}(\eta, \rho, 0)]\,d\eta. \tag{97}$$

By Eqs. (70)

$$T_{zz}(\eta, \rho, \zeta) = \frac{8}{\pi^2}\int_0^\infty \left[g\left(\frac{\gamma}{\eta}\right) F_4(\gamma, \rho, \alpha) + g^2\left(\frac{\gamma}{\eta}\right) F_5(\gamma, \rho, \alpha)\right]\cos\gamma\zeta\,d\gamma, \tag{98}$$

where

$$F_4(\gamma, \rho, \alpha) \equiv -\frac{2\alpha\gamma^2}{\Delta(\gamma)}\{[(\alpha+2)\, f_2(\gamma) + f_4(\gamma)]\; G(\gamma, \rho) + K(\gamma)\, f_2(\gamma)\; G'(\gamma, \rho)\};$$

$$F_5(\gamma, \rho, \alpha) \equiv -\frac{\gamma^2}{\Delta(\gamma)}\Big[\{[2K(\gamma) - \gamma^2 - 4]\, f_3(\gamma) + 4\alpha f_2(\gamma)\}\; G(\gamma, \rho) + K^2(\gamma)\, f_3(\gamma)\; G'(\gamma, \rho)\Big]. \tag{99}$$

The functions K, g, Δ, f_2, f_3, f_4, G, and G' are defined by Eqs. (55), (58), (61), and (68). Note that Poisson's ratio is involved in Eqs. (98) through the parameter α explicitly [see Eqs. (12)] and also implicitly through Δ and f_4.

In order to evaluate the integral in Eqs. (98), we used much the same procedure as in B above. To this end, define

$$F_4^0(\gamma, \rho, \alpha) \equiv \rho^{-\frac{1}{2}} e^{-\gamma R}(a_{41}\gamma + a_{40});$$

$$F_5^0(\gamma, \rho, \alpha) \equiv \rho^{-\frac{1}{2}} e^{-\gamma R}(a_{53}\gamma^3 + a_{52}\gamma^2 + a_{51}\gamma + a_{50}), \tag{100}$$

with

$$R \equiv \rho - 1, \tag{101}$$

and

$$a_{41}(\rho, \alpha) \equiv -2\alpha R;$$

$$a_{40}(\rho, \alpha) \equiv 2\alpha[1 + \alpha R + g_1(\rho) - \rho g_1'(\rho)];$$

$$a_{53}(\rho, \alpha) \equiv R;$$

$$a_{52}(\rho, \alpha) \equiv -1 + (1-\alpha)\, R - g_1(\rho) + \rho g_1'(\rho);$$

$$a_{51}(\rho, \alpha) \equiv \alpha - 1 + \tfrac{1}{8}(8\alpha^2 - 16\alpha + 47)\, R + (\alpha - 2)\, g_1(\rho) + (1-\alpha)\, \rho g_1'(\rho) - g_2(\rho) + \rho g_2'(\rho);$$

$$\begin{aligned} a_{50}(\rho, \alpha) \equiv\; & \tfrac{1}{2}(-2\alpha^2 + 12\alpha - 15) + \tfrac{1}{4}(-4\alpha^3 + 12\alpha^2 - 3\alpha - 15)\, R \\ & + \tfrac{1}{8}(-8\alpha^2 + 24\alpha - 55)\, g_1(\rho) + \tfrac{1}{8}(8\alpha^2 - 16\alpha + 47)\, \rho g_1'(\rho) \\ & + (\alpha - 2)\, g_2(\rho) + (1-\alpha)\, \rho g_2'(\rho) - g_3(\rho) + \rho g_3'(\rho). \end{aligned} \tag{102}$$

In Eqs. (102) the $g_i(\rho)$ and $g_i^{\prime}(\rho)$ are coefficients in the asymptotic expansions of G and G', and are given in Appendix B, Sect. 4.

By referring to Appendix B, Sect. 10, we see that F_4^0 and F_5^0, defined in Eqs. (100), are the leading terms in the asymptotic expansions of F_4 and F_5, respectively. Therefore, the functions

$$\tilde{F}_4(\gamma, \rho, \alpha) \equiv F_4(\gamma, \rho, \alpha) - F_4^0(\gamma, \rho, \alpha);$$

$$\tilde{F}_5(\gamma, \rho, \alpha) \equiv F_5(\gamma, \rho, \alpha) - F_5^0(\gamma, \rho, \alpha) \tag{103}$$

are much smaller than F_4 and F_5 at large values of γ, and are much easier to integrate numerically. Consequently, we write

$$T_{zz}(\eta, \rho, 0) = T_{zz}^0(\eta, \rho, 0) + \tilde{T}_{zz}(\eta, \rho, 0, \infty) \tag{104}$$

with

$$T_{zz}^0(\eta, \rho, 0) \equiv \frac{8}{\pi^2} \int_0^\infty \left[g\left(\frac{\gamma}{\eta}\right) F_4^0(\gamma, \rho, \alpha) + g^2\left(\frac{\gamma}{\eta}\right) F_5^0(\gamma, \rho, \alpha) \right] d\gamma;$$

$$\tilde{T}_{zz}(\eta, \rho, 0, \gamma_0) \equiv \frac{8}{\pi^2} \int_0^{\gamma_0} \left[g\left(\frac{\gamma}{\eta}\right) \tilde{F}_4(\gamma, \rho, \alpha) + g^2\left(\frac{\gamma}{\eta}\right) \tilde{F}_5(\gamma, \rho, \alpha) \right] d\gamma. \tag{105}$$

The integral expression for $T_{zz}^0(\eta, \rho, 0)$ can be expressed in terms of standard special functions, while the integral for $\tilde{T}_{zz}(\eta, \rho, 0, \infty)$ can be numerically integrated much more easily than the original integral for $T_{zz}(\eta, \rho, 0)$ given by Eqs. (98).

1. Evaluation of $T_{zz}^0(\eta, \rho, 0)$

The substitution of Eqs. (100) into the first of Eqs. (105) yields

$$T_{zz}^0(\eta, \rho, 0) = \frac{8}{\pi^2 \sqrt{\rho}} \left[\sum_{m=0}^{1} a_{4m} \int_0^\infty \gamma^m g\left(\frac{\gamma}{\eta}\right) e^{-\gamma R} d\gamma + \sum_{m=0}^{3} a_{5m} \int_0^\infty \gamma^m g^2\left(\frac{\gamma}{\eta}\right) e^{-\gamma R} d\gamma \right]. \tag{106}$$

Define

$$Q_{mn}(x) \equiv \frac{1}{m!} \int_0^\infty t^m g^n\left(\frac{t}{x}\right) e^{-t} dt, \qquad m, n = 0, 1, 2, 3, \ldots . \tag{107}$$

Then, by a change in the dummy variable, it is easily shown that

$$\int_0^\infty \gamma^m g^n\left(\frac{\gamma}{\eta}\right) e^{-\gamma R}\, d\gamma = \frac{m!}{R^{m+1}} Q_{mn}(\eta R). \tag{108}$$

Differentiating Eqs. (108) with respect to R and η yields the recurrence relations

$$Q_{m+1,n}(x) = Q_{mn}(x) - \frac{x}{m+1} Q'_{mn}(x)$$

and

$$Q_{m,n+1}(x) = Q_{mn}(x) - \frac{x}{2n} Q'_{mn}(x), \tag{109}$$

respectively.* We have from Ref. 21, p. 135, that

$$\begin{aligned} Q_{01}(x) &= x[\sin x\, \mathrm{Ci}(x) - \cos x\, \mathrm{si}(x)]; \\ Q_{11}(x) &= -x^2[\cos x\, \mathrm{Ci}(x) + \sin x\, \mathrm{si}(x)], \end{aligned} \tag{110}$$

where si(x) and Ci(x) are the sine and cosine integrals, respectively, defined by**

$$\mathrm{si}(x) \equiv -\int_x^\infty \frac{\sin t}{t}\, dt;$$

$$\mathrm{Ci}(x) \equiv -\int_x^\infty \frac{\cos t}{t}\, dt. \tag{111}$$

Using Eqs. (109), (110), and (111), we find that

---

*Although the information will not be used here, it is perhaps interesting to note the following properties of $Q_{mn}(x)$:
(a) $Q_{m0}(x) = 1$ by the definition of the gamma function;
(b) $Q_{2n,n}(x) = Q_{2n-1,n+1}(x)$ for $\eta > 0$, which can be proven by manipulation of Eqs. (107); and (c) $(m+2)!Q_{m+2,n}(x) = m!x^2[Q_{m,n-1}(x) - Q_{mn}(x)]$.

**See, for instance, Ref. 15, p. 145.

$$Q_{02}(x) = \frac{1}{2}[Q_{01}(x) + Q_{11}(x)];$$

$$Q_{12}(x) = \frac{1}{2}x^2[1 - Q_{01}(x)];$$

$$Q_{22}(x) = \frac{1}{4}x^2[Q_{01}(x) - Q_{11}(x)];$$

$$Q_{32}(x) = \frac{1}{6}x^2[Q_{11}(x) - Q_{12}(x)]. \tag{112}$$

Figure 21 is a plot of the $Q_{mn}(x)$ in Eqs. (110) and (112). Note that, by Eqs. (108) and (58),

$$Q_{mn}(0) = 0, \qquad n > 0, \tag{113}$$

and by the asymptotic expansions discussed in Appendix B, Sect. 11,

$$Q_{mn}(\infty) = 1. \tag{114}$$

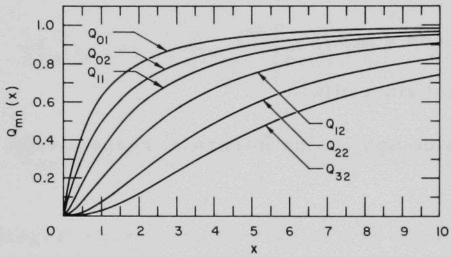

Fig. 21

The Function $Q_{mn}(x)$ for Various m and n

Substituting Eqs. (108) into Eq. (106), we arrive at

$$T^0_{zz}(\eta, \rho, 0) = \frac{8}{\pi^2\sqrt{\rho}}\left[\sum_{m=0}^{1} \frac{m!}{R^{m+1}} a_{4m}(\rho, \alpha)\, Q_{m1}(\eta R) + \sum_{m=0}^{3} \frac{m!}{R^{m+1}} a_{5m}(\rho, \alpha)\, Q_{m2}(\eta R)\right], \tag{115}$$

with the a_{4m} and a_{5m} being given by Eqs. (102), and the Q_{mn} by Eqs. (110) and (112). The computer evaluation of the $Q_{mn}(x)$ for large x is discussed in Appendix B, Sect. 11.

2. Evaluation of $\tilde{T}_{zz}(\eta, \rho, 0, \infty)$

Since computer integration over an infinite range is impossible, the approximation

$$\tilde{T}_{zz}(\eta, \rho, 0, \infty) \approx \tilde{T}_{zz}[\eta, \rho, 0, \gamma_0(\rho)] \tag{116}$$

will be made [see the second of Eqs. (105)]. It is apparent from (103), (100) and (B-51) that the functions $\widetilde{F}_4(\gamma, \rho, \alpha)$ and $\widetilde{F}_5(\gamma, \rho, \alpha)$ which appear in the integrand of $\widetilde{T}_{zz}$ have the asymptotic expansions

$$\widetilde{F}_j(\gamma, \rho, \alpha) \sim e^{-\gamma(\rho-1)} \sum_{m=1} b_{jm}(\rho, \alpha)\, \gamma^{-m}, \qquad j = 4, 5. \tag{117}$$

Unless ρ is very close to unity, the exponential factor will cause $\widetilde{F}_4$ and $\widetilde{F}_5$ to go to zero very rapidly with increasing γ. Therefore,* the approximation (116) is a very good one. For $\rho \geq 2$, γ_0 was taken to be ten; for $1 \leq \rho < 2$, $\gamma_0(\rho)$ was taken large enough to make $\widetilde{F}_4(\gamma_0, \rho, \alpha)$ and $F_5(\gamma_0, \rho, \alpha)$ negligible. The integrand was evaluated at γ intervals of 0.1 and Simpson's rule applied to these values in the computer program for $\widetilde{T}_{zz}(\eta, \rho, 0, \gamma_0)$.

D. Reformulation of $T_{zz}(\eta, 1, \zeta)$, $T_{\theta\theta}(\eta, 1, \zeta)$, and $T_{\theta z}(\eta, 1, \zeta)$ Needed for the Calculation of the Stress Distribution on the Hole

The nonzero stresses on the hole for the pure shear problem are $\tau^S_{zz}(1, \theta, \zeta)$, $\tau^S_{\theta\theta}(1, \theta, \zeta)$, and $\tau^S_{\theta z}(1, \theta, \zeta)$. By Eqs. (28) and (27) we have

$$\tau^S_{zz}(1, \theta, \zeta) = \tau^P_{zz}(1, \theta) + \tau^R_{zz}(1, \theta, \zeta) = 2(\alpha - 2)\, \tau_S \cos 2\theta + \tau^R_{zz}(1, \theta, \zeta);$$

$$\tau^S_{\theta\theta}(1, \theta, \zeta) = \tau^P_{\theta\theta}(1, \theta) + \tau^R_{\theta\theta}(1, \theta, \zeta) = -4\tau_S \cos 2\theta + \tau^R_{\theta\theta}(1, \theta, \zeta);$$

$$\tau^S_{\theta z}(1, \theta, \zeta) = \tau^P_{\theta z}(1, \theta) + \tau^R_{\theta z}(1, \theta, \zeta) = \tau^R_{\theta z}(1, \theta, \zeta). \tag{118}$$

By definition

$$\tau^R_{DD} \equiv \tau^R_{rr} + \tau^R_{\theta\theta} + \tau^R_{zz}, \tag{119}$$

so that

$$\tau^R_{\theta\theta}(1, \theta, \zeta) = \tau^R_{DD}(1, \theta, \zeta) - \tau^R_{zz}(1, \theta, \zeta), \tag{120}$$

since $\tau^R_{rr}(1, \theta, \zeta)$ is zero by boundary condition (29). As the expression for τ^R_{DD} is somewhat simpler than the expression for $\tau^R_{\theta\theta}$, it was decided to calculate $\tau^R_{zz}(1, \theta, \zeta)$, $\tau^R_{DD}(1, \theta, \zeta)$, and $\tau^R_{\theta z}(1, \theta, \zeta)$, and then to find $\tau^R_{\theta\theta}(1, \theta, \zeta)$ from the first two of these by means of Eqs. (120).

From Eqs. (69) we have

---

*Note that by Eqs. (58) $g(\gamma/\eta) < 1$ for all γ.

$$\frac{\tau^{R}_{zz}(1, \theta, \zeta)}{\tau_S \cos 2\theta} = \int_0^\infty D(\eta)\left[\frac{2}{\pi}\eta^3(1+\eta\zeta)\, e^{-\eta\zeta} + T_{zz}(\eta, 1, \zeta)\right] d\eta;$$

$$\frac{\tau^{R}_{DD}(1, \theta, \zeta)}{\tau_S \cos 2\theta} = (4 - \alpha)\int_0^\infty D(\eta)\left[\frac{2}{\pi}\eta^3 e^{-\eta\zeta} + T_{DD}(\eta, 1, \zeta)\right] d\eta;$$

$$\frac{\tau^{R}_{\theta z}(1, \theta, \zeta)}{\tau_S \sin 2\theta} = \int_0^\infty D(\eta)\left[\frac{4}{\pi}\eta^3\zeta\, e^{-\eta\zeta} + T_{\theta z}(\eta, 1, \zeta)\right] d\eta, \qquad (121)$$

in which the Wronskian formula*

$$\Omega_2(\eta, 1) = 2/\pi\eta \qquad (122)$$

has been used. Referring to Eqs. (70) we see that

$$T_{zz}(\eta, 1, \zeta) = \frac{8}{\pi^2}\int_0^\infty \left[g\left(\frac{\gamma}{\eta}\right) F_4(\gamma, 1, \alpha) + g^2\left(\frac{\gamma}{\eta}\right) F_5(\gamma, 1, \alpha)\right] \cos\gamma\zeta \, d\gamma;$$

$$T_{DD}(\eta, 1, \zeta) = \frac{8}{\pi^2}\int_0^\infty \left[g\left(\frac{\gamma}{\eta}\right) F_6(\gamma, 1, \alpha) + g^2\left(\frac{\gamma}{\eta}\right) F_7(\gamma, 1, \alpha)\right] \cos\gamma\zeta \, d\gamma;$$

$$T_{\theta z}(\eta, 1, \zeta) = \frac{8}{\pi^2}\int_0^\infty \left[g\left(\frac{\gamma}{\eta}\right) F_8(\gamma, 1, \alpha) + g^2\left(\frac{\gamma}{\eta}\right) F_9(\gamma, 1, \alpha)\right] \frac{\sin\gamma\zeta}{\gamma}\, d\gamma, \qquad (123)$$

where $g(x)$ is defined by Eqs. (58), $F_4(\gamma, \rho, \alpha)$ and $F_5(\gamma, \rho, \alpha)$ are defined by Eqs. (99), and

$$F_6(\gamma, \rho, \alpha) \equiv -\frac{2\alpha\gamma^2 f_2(\gamma)}{\Delta(\gamma)} G(\gamma, \rho);$$

$$F_7(\gamma, \rho, \alpha) \equiv -\frac{\gamma^2 K(\gamma)\, f_3(\gamma)}{\Delta(\gamma)} G(\gamma, \rho);$$

$$F_8(\gamma, \rho, \alpha) \equiv \frac{\alpha\gamma^2}{\rho\Delta(\gamma)} \{4[\alpha f_2(\gamma) + f_4(\gamma)]\, G(\gamma, \rho) + K(\gamma)[-f_1(\gamma) + 4f_2(\gamma)]\, G'(\gamma, \rho)\};$$

$$F_9(\gamma, \rho, \alpha) \equiv \frac{2\gamma^2}{\rho\Delta(\gamma)} \{[-(\gamma^2+4)\, f_3(\gamma) + 4\alpha f_2(\gamma)]\, G(\gamma, \rho) + K^2(\gamma)[f_3(\gamma) - \alpha f_2(\gamma)]\, G'(\gamma, \rho)\}.$$

$$(124)$$

---

*See Eqs. (49) and Ref. 15, p. 79.

The functions K, Δ, f_2, f_3, f_4, G, and G' are defined in Eqs. (55), (61), and (68). Letting $\rho = 1$ in Eqs. (99) and (124) and using Eqs. (68) we find

$$F_4(\gamma, 1, \alpha) = -\frac{2\alpha\gamma^2}{\Delta(\gamma)} \{[\alpha + 2 + K(\gamma)]\, f_2(\gamma) + f_4(\gamma)\};$$

$$F_5(\gamma, 1, \alpha) = -\frac{\gamma^2}{\Delta(\gamma)} \{4\alpha f_2(\gamma) + [K^2(\gamma) + 2K(\gamma) - \gamma^2 - 4]\, f_3(\gamma)\};$$

$$F_6(\gamma, 1, \alpha) = -\frac{2\alpha\gamma^2}{\Delta(\gamma)} f_2(\gamma);$$

$$F_7(\gamma, 1, \alpha) = -\frac{\gamma^2}{\Delta(\gamma)} K(\gamma)\, f_3(\gamma);$$

$$F_8(\gamma, 1, \alpha) = \frac{\alpha\gamma^2}{\Delta(\gamma)} \{-K(\gamma)\, f_1(\gamma) + 4[\alpha + K(\gamma)]\, f_2(\gamma) + 4f_4(\gamma)\};$$

$$F_9(\gamma, 1, \alpha) = \frac{2\gamma^2}{\Delta(\gamma)} \{\alpha[4 - K^2(\gamma)]\, f_2(\gamma) + [K^2(\gamma) - \gamma^2 - 4]\, f_3(\gamma)\}. \tag{125}$$

Proceeding as in Sects. B and C above, we define

$$\tilde{F}_j(\gamma, 1, \alpha) \equiv F_j(\gamma, 1, \alpha) - F_j^0(\gamma, 1, \alpha), \qquad j = 4, 5, \ldots 9, \tag{126}$$

where F_j^0 will be taken to be the terms in nonnegative powers of γ in the asymptotic expansion for the corresponding F_j. Therefore, referring to Appendix B, Sect. 12, the F_j^0 are defined as

$$F_4^0(\gamma, 1, \alpha) \equiv 2\alpha;$$

$$F_5^0(\gamma, 1, \alpha) \equiv -\gamma^2 + (\alpha - 1)\gamma + \tfrac{1}{2}(-2\alpha^2 + 12\alpha - 15);$$

$$F_6^0(\gamma, 1, \alpha) \equiv 2\alpha;$$

$$F_7^0(\gamma, 1, \alpha) \equiv -\gamma^2 + \tfrac{1}{2}(2\alpha - 1)\gamma + \tfrac{1}{4}(-4\alpha^2 + 6\alpha - 15);$$

$$F_8^0(\gamma, 1, \alpha) \equiv \alpha[\gamma + \tfrac{1}{2}(-8\alpha + 5)];$$

$$F_9^0(\gamma, 1, \alpha) \equiv 2(\alpha - 1)\gamma^2 + (-2\alpha^2 + 3\alpha)\gamma + \tfrac{1}{2}(4\alpha^3 - 10\alpha^2 - 3\alpha). \tag{127}$$

Note that the first two of Eqs. (127) are equivalent to letting $\rho = 1$ in Eqs. (100).

Substituting Eqs. (126) into Eqs. (123), we have

$$T_{zz}(\eta, 1, \zeta) = T^0_{zz}(\eta, 1, \zeta) + \tilde{T}_{zz}(\eta, 1, \zeta, \infty);$$

$$T_{DD}(\eta, 1, \zeta) = T^0_{DD}(\eta, 1, \zeta) + \tilde{T}_{DD}(\eta, 1, \zeta, \infty)$$

$$T_{\theta z}(\eta, 1, \zeta) = T^0_{\theta z}(\eta, 1, \zeta) + \tilde{T}_{\theta z}(\eta, 1, \zeta, \infty), \tag{128}$$

with

$$T^0_{zz}(\eta, 1, \zeta) \equiv \frac{8}{\pi^2}\int_0^\infty \left[g\left(\frac{\gamma}{\eta}\right) F^0_4(\gamma, 1, \alpha) + g^2\left(\frac{\gamma}{\eta}\right) F^0_5(\gamma, 1, \alpha)\right] \cos \gamma\zeta \, d\gamma;$$

$$T^0_{DD}(\eta, 1, \zeta) \equiv \frac{8}{\pi^2}\int_0^\infty \left[g\left(\frac{\gamma}{\eta}\right) F^0_6(\gamma, 1, \alpha) + g^2\left(\frac{\gamma}{\eta}\right) F^0_7(\gamma, 1, \alpha)\right] \cos \gamma\zeta \, d\gamma;$$

$$T^0_{\theta z}(\eta, 1, \zeta) \equiv \frac{8}{\pi^2}\int_0^\infty \left[g\left(\frac{\gamma}{\eta}\right) F^0_8(\gamma, 1, \alpha) + g^2\left(\frac{\gamma}{\eta}\right) F^0_9(\gamma, 1, \alpha)\right] \frac{\sin \gamma\zeta}{\gamma} \, d\gamma, \tag{129}$$

and

$$\tilde{T}_{zz}(\eta, 1, \zeta, \gamma_0) \equiv \frac{8}{\pi^2}\int_0^{\gamma_0} \left[g\left(\frac{\gamma}{\eta}\right) \tilde{F}_4(\gamma, 1, \alpha) + g^2\left(\frac{\gamma}{\eta}\right) \tilde{F}_5(\gamma, 1, \alpha)\right] \cos \gamma\zeta \, d\gamma;$$

$$\tilde{T}_{DD}(\eta, 1, \zeta, \gamma_0) \equiv \frac{8}{\pi^2}\int_0^{\gamma_0} \left[g\left(\frac{\gamma}{\eta}\right) \tilde{F}_6(\gamma, 1, \alpha) + g^2\left(\frac{\gamma}{\eta}\right) \tilde{F}_7(\gamma, 1, \alpha)\right] \cos \gamma\zeta \, d\gamma;$$

$$\tilde{T}_{\theta z}(\eta, 1, \zeta, \gamma_0) \equiv \frac{8}{\pi^2}\int_0^{\gamma_0} \left[g\left(\frac{\gamma}{\eta}\right) \tilde{F}_8(\gamma, 1, \alpha) + g^2\left(\frac{\gamma}{\eta}\right) \tilde{F}_9(\gamma, 1, \alpha)\right] \frac{\sin \gamma\zeta}{\gamma} \, d\gamma. \tag{130}$$

The integrals in Eqs. (129) can be expressed in terms of standard functions. Since by Appendix B, Sect. 13, the $\tilde{F}_j$ are all proportional to γ^{-1} for large γ, the integrals in Eqs. (130) are much easier to evaluate by numerical techniques than the original integrals in Eqs. (123).

1. Evaluation of $T^0_{zz}(\eta, 1, \zeta)$, $T^0_{DD}(\eta, 1, \zeta)$, and $T^0_{\theta z}(\eta, 1, \zeta)$

Define the auxiliary functions

$$S_{mn}(x) \equiv \frac{1}{m!}\int_0^{\infty} t^m g^n\left(\frac{t}{x}\right)\sin\left(t - \frac{m\pi}{2}\right)dt; \qquad m = 0, 1, 2, \ldots 2n - 1;$$

$$n = 1, 2, 3, \ldots \tag{131}$$

and

$$C_{mn}(x) \equiv \frac{2}{\pi}\int_0^{\infty} t^m g^n\left(\frac{t}{x}\right)\cos\left(t - \frac{m\pi}{2}\right)dt; \qquad m = -1, 0, 1, 2, \ldots 2n - 1;$$

$$n = 1, 2, 3, \ldots . \tag{132}$$

The factor $1/m!$ has been included in the definition of $S_{mn}(x)$ as it was in that of $Q_{mn}(x)$ [see Eqs. (107)] since thereby $S_{mn}(x) \to 1$ and $Q_{mn}(x) \to 1$ as $x \to \infty$, as is shown in Appendix B, Sects. 11 and 14. Including the same factor in the definition of $C_{mn}(x)$ would create difficulties, however, since m! is not defined for $m = -1$, whereas the integral in Eqs. (132) does exist for this value of m. Moreover, $C_{mn}(x)$ does not possess an asymptotic expansion of the same type as $S_{mn}(x)$ and $Q_{mn}(x)$, whose expansions are very similar in appearance, and the inclusion of a $1/m!$ factor would not "normalize" the large-argument behavior of C_{mn}. Note that, for both S_{mn} and C_{mn}, m must be less than $2n - 1$ for the integrals to exist.

From the definitions (131) and (132) and integration by parts, it can be shown that the functions $S_{mn}(x)$ satisfy the same recurrence relations (109) as $Q_{mn}(x)$, whereas the $C_{mn}(x)$ satisfy the recurrence relations

$$C_{m+1,n}(x) = (m+1)\,C_{mn}(x) - xC'_{mn}(x);$$

$$C_{m,n+1}(x) = C_{mn}(x) - \frac{x}{2n}\,C'_{mn}(x). \tag{133}$$

By changing the dummy variables in Eqs. (131) and (132), it is easily shown that

$$\int_0^{\infty} \gamma^m g^n\left(\frac{\gamma}{\eta}\right)\left[\cos\frac{m\pi}{2}\sin\gamma\zeta - \sin\frac{m\pi}{2}\cos\gamma\zeta\right]d\gamma = \frac{m!}{\zeta^{m+1}}\,S_{mn}(\eta\zeta);$$

$$\int_0^{\infty} \gamma^m g^n\left(\frac{\gamma}{\eta}\right)\left[\sin\frac{m\pi}{2}\sin\gamma\zeta + \cos\frac{m\pi}{2}\cos\gamma\zeta\right]d\gamma = \frac{\pi}{2\zeta^{m+1}}\,C_{mn}(\eta\zeta). \tag{134}$$

By Eqs. (129), (127), and (134) we have

$$T^0_{zz}(\eta, 1, \zeta) = \frac{4}{\pi}\left[\frac{2\alpha}{\zeta} C_{01}(\eta\zeta) + \frac{1}{\zeta^3} C_{22}(\eta\zeta) + \frac{(-2\alpha^2 + 12\alpha - 15)}{2\zeta} C_{02}(\eta\zeta)\right] + \frac{8(-\alpha+1)}{\pi^2\zeta^2} S_{12}(\eta\zeta);$$

$$T^0_{DD}(\eta, 1, \zeta) = \frac{4}{\pi}\left[\frac{2\alpha}{\zeta} C_{01}(\eta\zeta) + \frac{1}{\zeta^3} C_{22}(\eta\zeta) + \frac{(-4\alpha^2 + 6\alpha - 15)}{4\zeta} C_{02}(\eta\zeta)\right] + \frac{4(-2\alpha+1)}{\pi^2\zeta^2} S_{12}(\eta\zeta);$$

$$T^0_{\theta z}(\eta, 1, \zeta) = \frac{4}{\pi}\left[\frac{1}{2}(8\alpha^2 - 5\alpha) C_{-1,1}(\eta\zeta) + \frac{2(\alpha-1)}{\zeta^2} C_{12}(\eta\zeta) + \frac{1}{2}(-4\alpha^3 + 10\alpha^2 + 3\alpha) C_{-1,2}(\eta\zeta)\right]$$
$$+ \frac{8\alpha}{\pi^2\zeta}\left[S_{01}(\eta\zeta) + (-2\alpha+3) S_{02}(\eta\zeta)\right]. \qquad (135)$$

The integrals for $C_{-1,1}(x)$ and $S_{01}(x)$ are given in Ref. 21, p. 65. The expressions for the other combinations of m and n that are needed can be obtained from use of the recurrence relations for C_{mn} and S_{mn}. As a result, we have

$$C_{-1,1}(x) = e^{-x} - 1;$$
$$C_{01}(x) = x\, e^{-x};$$
$$C_{-1,2}(x) = \tfrac{1}{2}(x+2)\, e^{-x} - 1;$$
$$C_{02}(x) = \tfrac{1}{2}x(x+1)\, e^{-x};$$
$$C_{12}(x) = \tfrac{1}{2}x^3\, e^{-x};$$
$$C_{22}(x) = \tfrac{1}{2}x^3(x-1)\, e^{-x}, \qquad (136)$$

and

$$S_{01}(x) = \tfrac{1}{2}x[e^{-x} E^*(x) + e^{x} E_1(x)];$$
$$S_{11}(x) = \tfrac{1}{2}x^2[e^{-x} E^*(x) - e^{x} E_1(x)];$$
$$S_{02}(x) = \tfrac{1}{2}[S_{01}(x) + S_{11}(x)];$$
$$S_{12}(x) = \tfrac{1}{2}x^2[S_{01}(x) - 1]. \qquad (137)$$

The exponential integrals E^* and E_1 are defined in Ref. 15, p. 143:

$$E^*(x) \equiv -\int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt; \qquad E_1(x) \equiv \int_{x}^{\infty} \frac{e^{-t}}{t}\, dt. \qquad (138)$$

Figures 22 and 23 are graphs of the S_{mn} and C_{mn} functions listed in Eqs. (136) and (137).

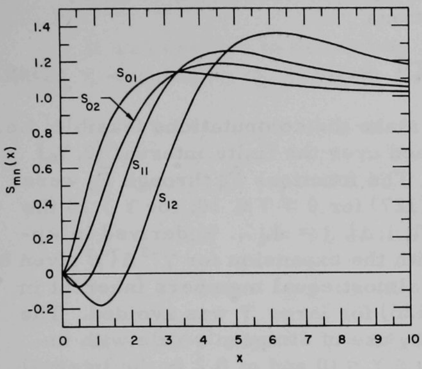

Fig. 22. The Function $S_{mn}(x)$ for Various m and n

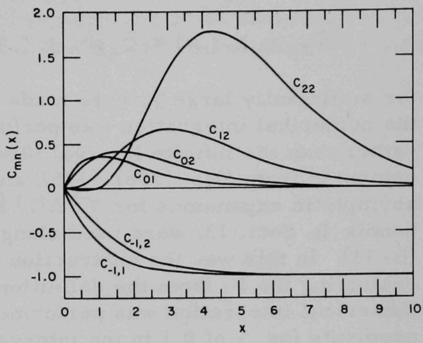

Fig. 23. The Function $C_{mn}(x)$ for Various m and n

Referring to the integrands in Eqs. (121), we find from Eqs. (128), (135), and (136) that

$$\frac{2}{\pi}\eta^3(1+\eta\zeta)\,e^{-\eta\zeta} + T_{zz}(\eta, 1, \zeta) = \frac{1}{\pi}[4\eta^4\zeta + (-2\alpha^2 + 12\alpha - 15)\,\eta^2\zeta$$

$$+ (-2\alpha^2 + 20\alpha - 15)\,\eta]\,e^{-\eta\zeta} + \frac{8(-\alpha+1)}{\pi^2\zeta^2}\,S_{12}(\eta\zeta) + \tilde{T}_{zz}(\eta, 1, \zeta, \infty);$$

$$\frac{2}{\pi}\eta^3\,e^{-\eta\zeta} + T_{DD}(\eta, 1, \zeta) = \frac{1}{\pi}[2\eta^4\zeta + \tfrac{1}{2}(-4\alpha^2 + 6\alpha - 15)\,\eta^2\zeta$$

$$+ \tfrac{1}{2}(-4\alpha^2 + 22\alpha - 15)\,\eta]\,e^{-\eta\zeta} + \frac{4(-2\alpha+1)}{\pi^2\zeta^2}\,S_{12}(\eta\zeta) + \tilde{T}_{DD}(\eta, 1, \zeta, \infty);$$

$$\frac{4}{\pi}\eta^3\zeta\,e^{-\eta\zeta} + T_{\theta z}(\eta, 1, \zeta) = \frac{\alpha}{\pi}\{[4\eta^3\zeta + (-4\alpha^2 + 10\alpha + 3)\,\eta\zeta]\,e^{-\eta\zeta}$$

$$+ 4(2\alpha^2 - 9\alpha + 1)(1 - e^{-\eta\zeta})\} + \frac{8\alpha}{\pi^2\zeta}[S_{01}(\eta\zeta) + (-2\alpha+3)\,S_{02}(\eta\zeta)] + \tilde{T}_{\theta z}(\eta, 1, \zeta, \infty). \tag{139}$$

Since adequate computer subroutines for the exponential integrals were not available, the $S_{mn}(x)$ were calculated from tabulated values of E^* and E_1.[22,23] For the asymptotic expansion of $S_{mn}(x)$, see Appendix B, Sect. 14. The terms in $e^{-\eta\zeta}$ in Eqs. (139) of course present no computational difficulties. The calculation of $\tilde{T}_{zz}$, $\tilde{T}_{DD}$, and $\tilde{T}_{\theta z}$ is discussed next.

2. Evaluation of $\tilde{T}_{zz}(\eta, 1, \zeta, \infty)$, $\tilde{T}_{DD}(\eta, 1, \zeta, \infty)$, and $\tilde{T}_{\theta z}(\eta, 1, \zeta, \infty)$

These integrals, defined in Eqs. (130), were numerically evaluated on the computer. The approximations

$$\tilde{T}_{zz}(\eta, 1, \zeta, \infty) \approx \tilde{T}_{zz}[\eta, 1, \zeta, \gamma_0(\zeta)] \quad \text{etc.,} \tag{140}$$

for sufficiently large γ_0 were made to make the computations feasible, i.e., the numerical integration was performed over the finite interval $(0, \gamma_0)$ rather than the infinite interval $(0, \infty)$. The functions $\tilde{F}_4$ through $\tilde{F}_9$ were computed from Eqs. (126), (125), and (127) for $0 \leq \gamma \leq 10$; for $\gamma > 10$ the asymptotic expansions for $\gamma^{-5}\Delta(\gamma)\,\tilde{F}_j(\gamma, 1, \alpha)$, $j = 4, \ldots 9$, derived in Appendix B, Sect. 13, were used along with the expansion for $\gamma^{-5}\Delta(\gamma)$ given by (B-33). In this way the subtraction of almost equal members inherent in evaluating the $\tilde{F}_j$ from the definition (126) for large γ was avoided. The numerical integration was performed by use of Simpson's rule with increments for γ of 0.1 in the interval $0 \leq \gamma \leq 10$ and of 0.2 in the interval $10 < \gamma \leq 100$. The contribution of the integration over the interval $100 < \gamma \leq 2000$ was calculated for a few values of ζ and found to be negligible compared to the integration over $0 \leq \gamma \leq 100$, so γ_0 in Eqs. (140) was taken to be 100.

E. Numerical Solution of the Integral Equation for $D(\eta)$

The numerical evaluation of the kernel $L(\xi, \eta)$ of the integral equation (65) has been discussed in Sect. B. In brief, for any combination of ξ and η the kernel L can be calculated by means of the computer through the use of Eqs. (78), (79), (82), (84), (94), and (95); accordingly, $L(\xi, \eta)$ will be considered to be a (numerically) known function.

For convenience we repeat the integral Eqs. (65):

$$D(\eta)\,\eta^6 Y(\eta) = \frac{8(2-\alpha)}{\pi} + \int_0^\infty D(\xi)\,L(\xi, \eta)\,d\xi. \tag{141}$$

Since by Eqs. (58)

$$g(\gamma/\eta) = 0 \text{ at } \eta = 0, \tag{142}$$

it follows from Eqs. (67) that

$$L(\xi, 0) = 0; \tag{143}$$

from Eqs. (66) and Ref. 13 (or see Appendix A, Sect. 4), we have

$$\eta^6 Y(\eta) = 64/\pi^2 \text{ at } \eta = 0. \tag{144}$$

Hence, from Eqs. (141), (143), and (144)

$$D(0) = (2 - \alpha)\pi/8. \tag{145}$$

It was decided to determine $D(\eta)$ numerically at an additional 75 values of η, denoted by η_j, $j = 1, 2, \ldots, 75$. The η_j were taken to be as follows:

$$\begin{aligned}
\eta_j &= 0.25j, & j &= 1, 2, \ldots 20; \\
\eta_j &= 0.5j - 5, & j &= 21, 22, \ldots 30; \\
\eta_j &= j - 20, & j &= 31, 32, \ldots 40; \\
\eta_j &= 5j - 180, & j &= 41, 42, \ldots 56; \\
\eta_j &= 100j - 5500, & j &= 57, 58, \ldots 75;
\end{aligned} \tag{146}$$

in other words, the interval (0, 5) was covered in steps of 0.25, the interval (5, 10) in steps of 0.5, the interval (10, 20) in steps of 1, the interval (20, 100) in steps of 5, and the interval (100, 2000) in steps of 100.

Simpson's rule was applied to evaluate the integral $\int_0^\infty D(\xi)\, L(\xi, \eta_j)\, d\xi$ in Eqs. (141), i.e., the integral was replaced by a weighted sum of the integrand evaluated at various values of ξ. These values of ξ were taken to be the same as the values of η given in (146). Consequently, Eqs. (141) was approximated by

$$D(\eta_j)\, \eta_j^6 Y(\eta_j) = \frac{8(2 - \alpha)}{\pi} + \sum_{i=1}^{75} w(\xi_i)\, D(\xi_i)\, L(\xi_i, \eta_j), \qquad j = 1, 2, 3, \ldots 75, \tag{147}$$

where the $w(\xi_i)$ are the weighting factors appropriate to Simpson's rule. Eqs. (147) represent 75 simultaneous algebraic equations in the 75 unknowns $D(\eta_j)$. The solution of these 75 equations is a routine computer job, particularly as the diagonal terms of the resulting matrix-inversion problem are large compared to the off-diagonal terms.

The function $Y(\eta_j)$ appearing in Eqs. (147) and defined by Eqs. (66) was computed for $0 < \eta_j \leq 10$ using (A-8) along with the library subroutines for the appropriate Bessel functions; for $\eta_j > 10$ the asymptotic expansion given in Appendix B, Sect. 2 was employed. Figure 24 is a graph of $\eta^2 Y(\eta)$.

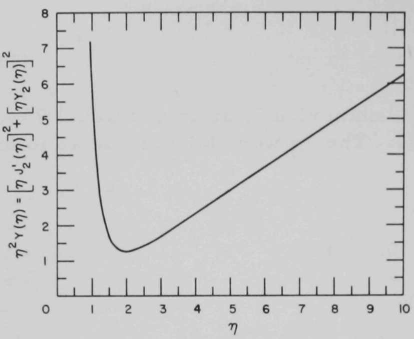

Fig. 24. The Function $\eta^2 Y(\eta)$

The computer solution of the integral equation, which included the evaluation of $L(\xi, \eta)$ of (78) for 2850 combinations of (ξ, η), the evaluation of $Y(\eta)$ at 75 values of η, and the solution of the resultant 75 simultaneous Eqs. (147) took slightly less than half an hour with the IBM-704 for each value of Poisson's ratio. An initial attempt to solve the integral equation through direct evaluation of the definition (67) of L took more computer time than this for only 27 values of η_j (or 378 combinations of ξ_i and η_j). Moreover, the values of $L(\xi, \eta)$ obtained using Eqs. (67) were less accurate than those obtained by determining L in the way discussed in Sect. B above.

Figures 2 and 3 are plots of $D(\eta)$ and $\eta^4 D(\eta)$, respectively, for Poisson's ratios of 1/4 and 1/2. From Fig. 3 we see that

$$D(\eta) \sim c(\nu)\, \eta^{-4} \tag{148}$$

with

$$c(\tfrac{1}{4}) = 0.322;$$

$$c(\tfrac{1}{2}) = 1.260 \tag{149}$$

approximately. The slight deviations away from this behavior for large η are probably due to truncating the numerical integration over η at 2000.

F. Numerical Check of the Boundary Condition on $\tau_{zz}(\rho, \theta, 0)$

With reference to the formulation of the pure shear problem, the only boundary condition (20) which is not automatically satisfied for any sufficiently well-behaved function $D(\eta)$ is the condition that the normal stress on the surface z = 0 must vanish. Expressed in terms of the residual problem this becomes the condition given by Eqs. (96).

From Eqs. (97) and (104) we have

$$\frac{\tau^R_{zz}(\rho, \theta, 0)}{\tau_S \cos 2\theta} = \int_0^\infty D(\eta)[\eta^4 \Omega_2(\eta, \rho) + T^0_{zz}(\eta, \rho, 0) + \tilde{T}_{zz}(\eta, \rho, 0, \infty)]\, d\eta, \tag{150}$$

with Ω_2 given by Eqs. (49), T^0_{zz} by Eqs. (115), and $\tilde{T}_{zz}$ by Eqs. (105). As discussed in Sect. C.2 above, $\tilde{T}_{zz}(\eta, \rho, 0, \infty)$ will be approximated by $\tilde{T}_{zz}(\eta, \rho, 0, \gamma_0)$. We can then write

$$\frac{\tau^R_{zz}(\rho, \theta, 0)}{\tau_S \cos 2\theta} \approx \int_0^{\epsilon_1} D(\eta)[\eta^4 \Omega_2(\eta, \rho) + T^0_{zz}(\eta, \rho, 0)]\, d\eta$$

$$+ \int_0^{\epsilon_2} D(\eta)\, \tilde{T}_{zz}(\eta, \rho, 0, \gamma_0)\, d\eta + \int_{\epsilon_1}^{\infty} D(\eta)\, \eta^4 \Omega_2(\eta, \rho)\, d\eta$$

$$+ \int_{\epsilon_1}^{\infty} D(\eta)\, T^0_{zz}(\eta, \rho, 0)\, d\eta + \int_{\epsilon_2}^{\infty} D(\eta)\, \tilde{T}_{zz}(\eta, \rho, 0, \gamma_0)\, d\eta, \tag{151}$$

where ϵ_1 and ϵ_2 may depend on ρ in general.

The values of ρ at which the boundary condition was checked are those listed in Tables I and II. Numerical integration based on Simpson's rule was used to compute the first two integrals in Eqs. (151). For all ρ values ϵ_2 was taken equal to 20; for $1 \le \rho \le 1.2$, a value of 100 was taken for ϵ_1; for $\rho > 1.2$, again a value of 20 was taken for ϵ_1. Thus we have

$$\int_0^{\epsilon_1} D(\eta)[\eta^4 \Omega_2(\eta, \rho) + T^0_{zz}(\eta, \rho, 0)]\, d\eta \approx \sum_{j=1}^{N(\rho)} w(\eta_j)\, D(\eta_j)[\eta_j^4 \Omega_2(\eta_j, \rho) + T^0_{zz}(\eta_j, \rho, 0)], \tag{152}$$

with

$$N(\rho) = 56, \quad 1 \le \rho \le 1.2; \qquad N(\rho) = 40, \quad \rho > 1.2.$$

The asymptotic expansion derived in Appendix B, Sect. 3, was used to compute $\Omega_2(\eta, \rho)$ for $\eta\rho > 20$. The η_j are those given by Eqs. (146), the $w(\eta_j)$ are the appropriate weighting factors for Simpson's rule, and the $D(\eta_j)$ are the results of the numerical solution of the integral equation. Similarly,

$$\int_0^{\epsilon_2} D(\eta)\, \tilde{T}_{zz}(\eta, \rho, 0, \gamma_0)\, d\eta \approx \sum_{j=1}^{40} w(\eta_j)\, D(\eta_j)\, \tilde{T}_{zz}(\eta_j, \rho, 0, \gamma_0). \tag{153}$$

As ρ increases, the period of the oscillations of the function $\Omega_2(\eta, \rho)$ becomes shorter, as may be seen in Fig. 25. The points η_j at which $D(\eta)$ were determined are too widely spaced to follow these oscillations, and the sum in Eqs. (152) is a very poor approximation of the integral when ρ is large. Therefore, for $\rho \ge 2$, the quantity $\eta^4 \Omega_2(\eta, \rho)$ was calculated at η

intervals of 0.01. The function $D(\eta)$ was estimated at the intermediate values of η in each of the intervals (η_{j-1}, η_{j+1}), $j = 1, 3, 5, \ldots 39$, by passing a parabola through the known values $D(\eta_{j-1})$, $D(\eta_j)$, and $D(\eta_{j+1})$. The trapezoidal rule was then applied to these small η intervals to evaluate $\int_0^{20} D(\eta)\, \eta^4 \Omega_2(\eta, \rho)\, d\eta$.

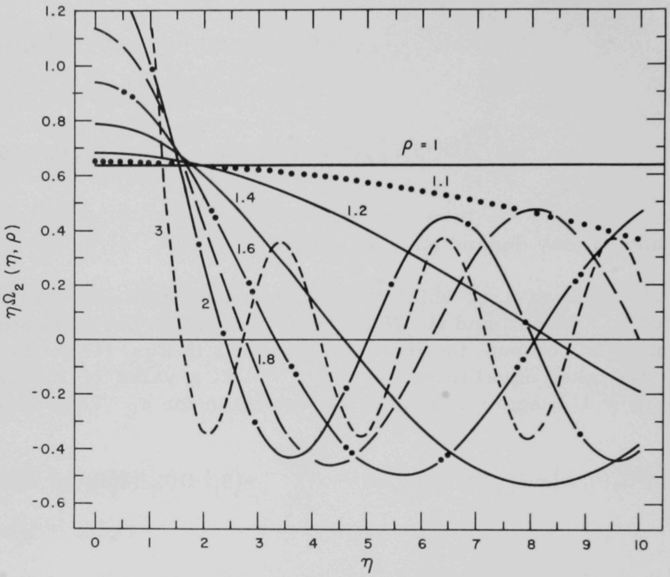

Fig. 25. $\eta\Omega_2(\eta, \rho)$ as a Function of η for Various Values of ρ

In the evaluation of the remaining three integrals in Eqs. (151), it was assumed that the asymptotic representation (148) for $D(\eta)$ was valid for $\eta > \epsilon_1$ and $\eta > \epsilon_2$. Since $\tilde{T}_{zz}$ approaches a constant value for large η, it is easily shown that

$$\int_{\epsilon_2}^{\infty} D(\eta)\, \tilde{T}_{zz}(\eta, \rho, 0, \gamma_0)\, d\eta = O(\epsilon_2^{-3}). \tag{154}$$

It was believed that the error introduced by neglecting this quantity would be less than the error inherent in the numerical integrations.

If $\epsilon_1 R$ is large enough,* then, for $\eta > \epsilon_1$, the quantities $Q_{mn}(\eta R)$ appearing in the representation (115) for T^0_{zz} can be replaced by their asymptotic expansions (see Appendix B, Sect. 11). In that case, since $D(\eta) \sim c(\nu)\,\eta^{-4}$ for $\eta > \epsilon_1$, we can show that

$$\int_{\epsilon_1}^{\infty} D(\eta)\, T^0_{zz}(\eta, \rho, 0)\, d\eta = O(\epsilon_1 R)^{-3}. \tag{155}$$

For most values of ρ at which the boundary condition was checked, the quantity $\epsilon_1 R$ was large enough so that a) the asymptotic expansion for the Q_{mn} were valid and b) the error introduced by neglecting the integral in Eqs. (155) would be no larger than the other errors in the numerical procedure. The two values of ρ that are an exception to this are $\rho = 1$ and $\rho = 1.02$. The estimation of the integral in Eqs. (155) for $\rho = 1$ will be discussed later. The integral was neglected for $\rho = 1.02$, although no estimate was available for the error in doing so. Note from Tables I and II that the deviations between the calculated and theoretical values of the boundary stress are greatest at $\rho = 1.02$. This may well be due, not to inaccuracy in the solution of the integral equation for $D(\eta)$, but to not taking ϵ_1 large enough in the stress calculation at $\rho = 1.02$.

Substituting the asymptotic expansion for $\Omega_2(\eta, \rho)$, derived in Appendix B, Sect. 3, into the third integral in Eqs. (151) and replacing $D(\eta)$ by its large argument behavior (148), we obtain

$$\int_{\epsilon_1}^{\infty} D(\eta)\, \eta^4 \Omega_2(\eta, \rho)\, d\eta \approx \frac{2c(\nu)}{\pi\sqrt{\rho}} \int_{\epsilon_1}^{\infty} \sum_{m=0} \frac{\omega_m(R)}{\rho^m \eta^{m+1}} \cos\left(\eta R - \frac{m\pi}{2}\right) d\eta \tag{156}$$

or, interchanging the order of integration and summation,

$$\int_{\epsilon_1}^{\infty} D(\eta)\, \eta^4 \Omega_2(\eta, \rho)\, d\eta \approx -\frac{2c(\nu)}{\pi\sqrt{\rho}} \sum_{m=0} \left(\frac{R}{\rho}\right)^m \frac{\omega_m(R)}{m!} W_m(\epsilon_1 R), \tag{157}$$

where

$$W_m(x) \equiv m! \int_{\infty}^{x} \frac{\cos\left(t - \frac{m\pi}{2}\right)}{t^{m+1}}\, dt, \qquad m = 0, 1, 2 \ldots . \tag{158}$$

From the definition (111) of the cosine integral, we have

$$W_0(x) = \mathrm{Ci}(x), \tag{159}$$

*Recall that $R \equiv \rho - 1$.

whereas in general

$$W_m(x) = \mathrm{Ci}(x) - \sum_{j=0}^{m-1} \frac{j!\,\sin\left(x - \frac{j\pi}{2}\right)}{x^{j+1}}, \qquad m = 1, 2, 3 \ldots . \tag{160}$$

The relations (160) may be verified by substituting Eqs. (158) for $W_m(x)$ and differentiating with respect to x. By means of tabulated values,[22] the contribution (157) was evaluated for the required values of ρ and added to the computer results for Eqs. (152) and (153). Only a few terms of the series in Eqs. (157) are needed since for small R the quantity R^m becomes very small as R increases, while for larger R, taking into consideration the values of ϵ_1 that were chosen, the $W_m(\epsilon_1 R)$ become very small as m increases. This is because the series subtracted from Ci(x) in Eqs. (160) is equal to the first m terms in the asymptotic expansion of Ci(x) (see Appendix B, Sect. 15).

At $\rho = 1$ the third and fourth integrals in Eqs. (151) must be considered together since each individually blows up* at this value of ρ. In Eqs. (115) for T^0_{zz} the terms with coefficients a_{53} and a_{52} are those which upon substitution into the fourth integral in Eqs. (151) produce nonzero results for $\rho = 1$; from Eqs. (157) and the footnote below, we see that in the asymptotic expansion of Ω_2 only the term corresponding to $m = 0$ will be of interest, since the other terms contribute nothing at $\rho = 1 (R = 0)$.

Thus, retaining only the term in ω_0 in the asymptotic expansion (B-14), and the terms in a_{53} and a_{52} in Eqs. (115), we will write, using Eqs. (148),

$$I(R, \epsilon_1) \equiv \int_{\epsilon_1}^{\infty} D(\eta)[\eta^4 \Omega_2(\eta, \rho) + T^0_{zz}(\eta, \rho, 0)]\, d\eta$$

$$\approx \frac{8c(\nu)}{\pi^2 \sqrt{\rho}} \int_{\epsilon_1}^{\infty} \left[\frac{\pi\omega_0(R)}{4\eta} \cos \eta R + \frac{6a_{53}(\rho, \alpha)\, Q_{32}(\eta R)}{\eta^4 R^4} + \frac{2a_{52}(\rho, \alpha)\, Q_{22}(\eta R)}{\eta^4 R^3}\right] d\eta \tag{161}$$

at $\rho = 1$ (or $R = 0$). By Eqs. (B-16), (102), (B-23), and (B-21)

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*For instance, the leading term in Eqs. (157) is $-\frac{2c}{\pi\sqrt{\rho}}\, \omega_0(R)\, W_0(\epsilon_1 R)$. By Eqs. (159), (B-16) this becomes $-\frac{2c}{\pi\sqrt{\rho}} \mathrm{Ci}(\epsilon_1 R)$, which blows up as $R \to 0$ since $\mathrm{Ci}(x) \to \log x$ as $x \to 0$.

$$\omega_0(R) = 1;$$

$$a_{53}(\rho, \alpha) = R;$$

$$a_{52}(\rho, \alpha) = -1 + O(R). \tag{162}$$

The terms of order R in a_{52} do not contribute anything to the integral in Eqs. (161) at $\rho = 1$ and so will be ignored. Substituting Eqs. (162) and the expressions for Q_{32} and Q_{22} from Eqs. (112) and (110) into Eqs. (161), we obtain

$$I(R, \epsilon_1) \approx \frac{4c(\nu)}{\pi^2\sqrt{1+R}} \int_{\epsilon_1}^{R} \left\{ \frac{\pi}{2} \frac{\cos \eta R}{\eta} - R \right.$$

$$- 3R[\cos \eta R \, Ci(\eta R) + \sin \eta R \, si(\eta R)]$$

$$\left. + \left(\eta^2 R - \frac{1}{\eta}\right)[\sin \eta R \, Ci(\eta R) - \cos \eta R \, si(\eta R)] \right\} d\eta \tag{163}$$

at R = 0. Using integration by parts we can show that

$$I(R, \epsilon_1) \approx \frac{4c(\nu)}{\pi^2\sqrt{1+R}} \left\{ -\frac{\pi}{2} Ci(\epsilon_1 R) + 2 \sin \epsilon_1 R \, Ci(\epsilon_1 R) \right.$$

$$- 2 \cos \epsilon_1 R \, si(\epsilon_1 R) - si(\epsilon_1 R) \, Ci(\epsilon_1 R) - 2 \int_{\epsilon_1 R}^{\infty} \frac{\sin t \, Ci(t)}{t} \, dt$$

$$\left. + \epsilon_1 R[\cos \epsilon_1 R \, Ci(\epsilon_1 R) + \sin \epsilon_1 R \, si(\epsilon_1 R)] \right\}, \tag{164}$$

which can be checked by differentiating with respect to ϵ_1. The integral* in Eqs. (164) is zero at R = 0; we have from Ref. 15, p. 146, that

$$si(x) = -\frac{\pi}{2} + O(x);$$

$$Ci(x) = \log x + O(1), \qquad \text{as } x \to 0. \tag{165}$$

Therefore, letting R = 0 in Eqs. (164) yields

$$I(0, \epsilon_1) \approx 4c(\nu)/\pi. \tag{166}$$

This quantity was added to the computer results for Eqs. (152) and (153) at $\rho = 1$.

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*See, for instance, Ref. 16, p. 161.

It is interesting to observe that if we let R go to zero in Eqs. (163) before performing the integration, the integrand, and therefore the integral, will be zero. Hence, there is a difference of π between a) the limit of the integral as R goes to zero and b) the integral of the limit as R goes to zero, i.e., denoting the integrand in Eqs. (163) by $f(\eta, R)$,

$$\lim_{R \to 0} \int_{\epsilon_1}^{\infty} f(\eta, R)\, d\eta = \pi;$$

$$\int_{\epsilon_1}^{\infty} \left[\lim_{R \to 0} f(\eta, R)\right] d\eta = 0. \tag{167}$$

This phenomenon is an indication of why it is difficult to get good numerical results for the τ_{zz} stress on $\zeta = 0$ near $\rho = 1$. The solution $D(\eta)$ of the integral equation is probably more accurate than can be concluded from the check of the boundary condition, since part of the deviation reported in Tables I and II is undoubtedly due to inaccuracies in the numerical computation of $\tau_{zz}(\rho, \theta, 0)$.

G. Numerical Calculations of the Stresses on the Surface of the Hole

The desired stresses are $\tau^S_{zz}(1, \theta, \zeta)$, $\tau^S_{\theta\theta}(1, \theta, \zeta)$, and $\tau^S_{\theta z}(1, \theta, \zeta)$. As discussed in Sect. D above, these stress distributions are easily determined from $\tau^R_{zz}(1, \theta, \zeta)$, $\tau^R_{DD}(1, \theta, \zeta)$, and $\tau^R_{\theta z}(1, \theta, \zeta)$. By Eqs. (121), (139), and (140), we can write

$$\frac{\tau^R_{zz}(1, \theta, \zeta)}{\tau_S \cos 2\theta} \approx \int_0^{\epsilon_1} D(\eta)\, T_1(\eta, \zeta)\, d\eta + \int_0^{\epsilon_2} D(\eta)[T_2(\eta, \zeta)$$

$$+ \tilde{T}_{zz}(\eta, 1, \zeta, \gamma_0)]\, d\eta + \int_{\epsilon_1}^{\infty} D(\eta)\, T_1(\eta, \zeta)\, d\eta$$

$$+ \int_{\epsilon_2}^{\infty} D(\eta)[T_2(\eta, \zeta) + \tilde{T}_{zz}(\eta, 1, \zeta, \gamma_0)]\, d\eta, \tag{168}$$

with

$$T_1(\eta, \zeta) \equiv \frac{1}{\pi}[4\eta^4\zeta + (-2\alpha^2 + 12\alpha - 15)\,\eta^2\zeta + (-2\alpha^2 + 20\alpha - 15)\,\eta]\, e^{-\eta\zeta};$$

$$T_2(\eta, \zeta) \equiv \frac{8(1 - \alpha)}{\pi^2\zeta^2}\, S_{12}(\eta\zeta), \tag{169}$$

where S_{12} is given in Eqs. (137). Similar relations hold for τ_{DD}^{R} and $\tau_{\theta z}^{R}$, but, since the procedure for evaluating all three stress distributions is the same, only τ_{zz}^{R} will be discussed in detail.

The stresses were evaluated for Poisson's ratios of 0.25 and 0.5 at

$$\zeta = 0,\ 0.01,\ 0.05,\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 1.0,\ 1.2,\ 1.6,\ 2.0,\ 3.0. \tag{170}$$

The integration limit ϵ_1 in Eqs. (168) was taken to be 1900 for $0 \le \zeta \le 0.5$ and to be 20 for $\zeta > 0.5$; for all ζ, ϵ_2 was taken to be 20. The first two integrals in Eqs. (168) were evaluated by use of Simpson's rule applied at the same η_j as listed in Eqs. (146). Thus

$$\int_0^{\epsilon_1} D(\eta)\, T_1(\eta, \zeta)\, d\eta \approx \sum_{j=1}^{N(\zeta)} w(\eta_j)\, D(\eta_j)\, T_1(\eta_j, \zeta);$$

$$N(\zeta) = 74, \quad 0 \le \zeta \le 0.5; \quad N(\zeta) = 40, \quad \zeta > 0.5; \tag{171}$$

and

$$\int_0^{\epsilon_2} D(\eta)[T_2(\eta, \zeta) + \tilde{T}_{zz}(\eta, 1, \zeta, \gamma_0)]\, d\eta$$

$$\approx \sum_{j=1}^{40} w(\eta_j)\, D(\eta_j)[T_2(\eta_j, \zeta) + \tilde{T}_{zz}(\eta_j, 1, \zeta, \gamma_0)]. \tag{172}$$

The $D(\eta_j)$ are the numerical solutions to the integral equation for the appropriate Poisson's ratio, and the $w(\eta_j)$ are the weighting factors for Simpson's rule.

The last integral in Eqs. (168) was neglected as it is $O(\epsilon_2^{-3})$ for $D(\eta)$ behaving as in Eqs. (148).

The third integral in Eqs. (168) is also negligible unless ζ is very small. The first term in $T_1(\eta, \zeta)$ defined in Eqs. (169) is the most significant for large η and small ζ. We will write

$$\int_{\epsilon_1}^{\infty} D(\eta)\, T_1(\eta, \zeta)\, d\eta \approx \frac{4}{\pi}\int_{\epsilon_1}^{\infty} D(\eta)\, \eta^4 \zeta\, e^{-\eta\zeta}\, d\eta \tag{173}$$

for η large and ζ small. Replacing $D(\eta)$ by $c(\nu)\ \eta^{-4}$ for $\epsilon_1 < \eta < \infty$ and performing the integration, we obtain

$$\int_{\epsilon_1}^{\infty} D(\eta)\ T_1(\eta, \zeta)\ d\eta \approx \frac{4c(\nu)}{\pi}\ e^{-\epsilon_1\zeta}. \tag{174}$$

This quantity is negligible for the selected combinations of ϵ_1 and ζ, except for $\zeta = 0$ when it is $4c/\pi$ [cf. Eq. (166)]. Note that the phenomenon discussed at the end of Sect. F above occurs here also, i.e., the integrand in Eqs. (173) is zero for $\zeta = 0$, so that performing the integration and then evaluating at $\zeta = 0$ gives a different result than letting $\zeta = 0$ and then doing the integration.

For τ_{DD}^{R} the result corresponding to Eqs. (174) is $\dfrac{2(4-\alpha)}{\pi}\ c(\nu)\ e^{-\epsilon_1\zeta}$, whereas for $\tau_{\theta z}^{R}$ it is $\dfrac{4\alpha}{\pi}\ c(\nu)\ \zeta E_1(\epsilon_1\zeta)$, where E_1 is the exponential integral defined in Eqs. (138).

The stress distributions on the surface of the hole are shown in Figs. 4, 5, and 6.

H. Numerical Calculation of the Normal Displacement on the Bounding Plane

The normal displacement on the plane $\zeta = 0$ for the pure shear problem is given by, using Eqs. (28) and (27),

$$u_z^S(\rho, \theta, 0) = u_z^P(\rho, \theta) + u_z^R(\rho, \theta, 0) = u_z^R(\rho, \theta, 0). \tag{175}$$

From the third of Eqs. (69) we have

$$\frac{2\mu u_z^R(\rho, \theta, 0)}{a\,\tau_S \cos 2\theta} = \int_0^{\infty} D(\eta)[-\alpha\eta^3\Omega_2(\eta, \rho) + U_z(\rho, \theta, 0)]\ d\eta. \tag{176}$$

Referring to Eqs. (70) we observe that*

$$U_z(\eta, \rho, 0) = 0, \tag{177}$$

so Eqs. (176) reduce to

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*See Ref. 14, p. 61.

$$\frac{2\mu u_z^R(\rho, \theta, 0)}{a\tau_S \cos 2\theta} = -\alpha \int_0^\infty D(\eta)\, \eta^3 \Omega_2(\eta, \rho)\, d\eta. \tag{178}$$

This integral was evaluated in much the same way as the integral in $\tau_{zz}^R(\rho, \theta, 0)$ containing the factor $\eta^4\Omega_2(\eta, \rho)$, the integration in Eqs. (178) being somewhat simpler due to the more rapid decay of its integrand with increasing η. As discussed in Sect. F above, the infinite interval of integration was truncated to the interval $[0, \epsilon_1(\rho)]$ and Simpson's rule applied to approximate the integral by a finite sum. For $\rho \geq 2$, when the oscillations of Ω_2 become too rapid to apply Simpson's rule directly to the values of $D(\eta)$ computed in the solution of the integral equation, parabolic interpolation was used to find $D(\eta)$ at η intervals of 0.01. Figure 7 is a graph of $u_z^S(\rho, \theta, 0)$.

III. APPENDICES

Appendix A. Bessel Functions and Related Functions

1. Definitions

The functions $J_n(x)$ and $Y_n(x)$ are the Bessel functions of the first and second kinds of order n, while $K_n(x)$ is the modified Bessel function of the second kind of order n.[13,15] The following expressions define related auxiliary functions which occur frequently in the text:

$$\left.\begin{aligned} J_2'(x) &\equiv \frac{dJ_2(x)}{dx}; \\ Y_2'(x) &\equiv \frac{dY_2(x)}{dx}; \\ K_2'(x) &\equiv \frac{dK_2(x)}{dx}. \end{aligned}\right\} \qquad \text{(A-1)}$$

$$K(x) \equiv xK_2'(x)/K_2(x). \qquad \text{(A-2)}$$

$$\left.\begin{aligned} \Omega_2(\gamma,\rho) &\equiv Y_2'(\gamma)\, J_2(\gamma\rho) - J_2'(\gamma)\, Y_2(\gamma\rho); \\ \Omega_2'(\gamma,\rho) &\equiv Y_2'(\gamma)\, J_2'(\gamma\rho) - J_2'(\gamma)\, Y_2'(\gamma\rho); \\ \Omega_2^0(\gamma,\rho) &\equiv Y_2(\gamma)\, J_2(\gamma\rho) - J_2(\gamma)\, Y_2(\gamma\rho). \end{aligned}\right\} \qquad \text{(A-3)}$$

$$Y(x) \equiv [J_2'(x)]^2 + [Y_2'(x)]^2. \qquad \text{(A-4)}$$

$$\left.\begin{aligned} G(\gamma,\rho) &\equiv K_2(\gamma\rho)/K_2(\gamma); \\ G'(\gamma,\rho) &\equiv \rho K_2'(\gamma\rho)/K_2'(\gamma). \end{aligned}\right\} \qquad \text{(A-5)}$$

$$\left.\begin{aligned} f_1(\gamma) &\equiv \gamma^2[\gamma^2 + 4 - K^2(\gamma)] + 3\alpha K^2(\gamma); \\ f_2(\gamma) &\equiv \gamma^2 - (\gamma^2 + 3)\, K(\gamma); \\ f_3(\gamma) &\equiv (\gamma^2 + 2)(\gamma^2 + 6) - 2\gamma^2 K(\gamma); \\ f_4(\gamma) &\equiv (\gamma^2 + 3)(\gamma^2 + 4) - \gamma^2 K(\gamma) - \alpha f_2(\gamma) + 3\alpha K(\gamma); \\ \Delta(\gamma) &\equiv [\gamma^2 + 4 - K^2(\gamma)]\, f_3(\gamma) + \alpha[-4\gamma^2 + 8(\gamma^2 + 3)\, K(\gamma) \\ &\qquad - \gamma^2 K^2(\gamma) - 6K^3(\gamma)]. \end{aligned}\right\} \qquad \text{(A-6)}$$

Figures 12, 13, 24, and 25 show K, Δ, Y, and Ω_2, respectively.

2. Recurrence Relations[13,15]

$$\left.\begin{aligned} xJ_2(x) &= 2J_1(x) - xJ_0(x); \\ xY_2(x) &= 2Y_1(x) - xY_0(x); \\ xK_2(x) &= 2K_1(x) + xK_0(x). \end{aligned}\right\} \quad \text{(A-7)}$$

$$\left.\begin{aligned} x^2J_2'(x) &= (x^2 - 4)\, J_1(x) + 2xJ_0(x); \\ x^2Y_2'(x) &= (x^2 - 4)\, Y_1(x) + 2xY_0(x); \\ x^2K_2'(x) &= -(x^2 + 4)\, K_1(x) - 2xK_0(x). \end{aligned}\right\} \quad \text{(A-8)}$$

3. Differentiation Formulas

$$\left.\begin{aligned} \frac{\partial f(\gamma\rho)}{\partial\rho} &= \gamma \frac{df(x)}{dx}\bigg|_{x = \gamma\rho}; \\ \frac{\partial^2 f(\gamma\rho)}{\partial\rho^2} &= \gamma^2 \frac{d^2f(x)}{dx^2}\bigg|_{x = \gamma\rho}. \end{aligned}\right\} \quad \text{(A-9)}$$

$$\left.\begin{aligned} \frac{dK_2(x)}{dx} &= K_2'(x); \\ \frac{d^2K_2(x)}{dx^2} &= \left(1 + \frac{4}{x^2}\right) K_2(x) - \frac{1}{x} K_2'(x); \\ \frac{d}{dx}\left[\frac{1}{x} K_2(x)\right] &= \frac{1}{x^2} K_2(x) + \frac{1}{x} K_2'(x); \\ \frac{d^2}{dx^2}\left[\frac{1}{x} K_2(x)\right] &= \left(\frac{1}{x} + \frac{6}{x^3}\right) K_2(x) - \frac{3}{x^2} K_2'(x); \\ \frac{dK_2'(x)}{dx} &= \left(1 + \frac{4}{x^2}\right) K_2(x) - \frac{1}{x} K_2'(x); \\ \frac{d^2K_2'(x)}{dx^2} &= -\left(\frac{1}{x} + \frac{12}{x^3}\right) K_2(x) + \left(1 + \frac{6}{x^2}\right) K_2'(x); \\ \frac{d}{dx}[xK_2'(x)] &= \left(x + \frac{4}{x}\right) K_2(x); \\ \frac{d^2}{dx^2}[xK_2'(x)] &= \left(1 - \frac{4}{x^2}\right) K_2(x) + \left(x + \frac{4}{x}\right) K_2'(x); \\ \frac{d}{dx}\left[\frac{1}{x} K_2'(x)\right] &= \left(\frac{1}{x} + \frac{4}{x^3}\right) K_2(x) - \frac{2}{x^2} K_2'(x). \end{aligned}\right\} \quad \text{(A-10)}$$

$$\left.\begin{aligned}
\frac{\partial \Omega_2(\gamma,\rho)}{\partial \rho} &= \gamma \Omega_2'(\gamma,\rho); \\
\frac{\partial^2 \Omega_2(\gamma,\rho)}{\partial \rho^2} &= \left(\frac{4}{\rho^2} - \gamma^2\right) \Omega_2(\gamma,\rho) - \frac{\gamma}{\rho} \Omega_2'(\gamma,\rho); \\
\frac{\partial \Omega_2'(\gamma,\rho)}{\partial \rho} &= \left(\frac{4}{\gamma\rho^2} - \gamma\right) \Omega_2(\gamma,\rho) - \frac{1}{\rho} \Omega_2'(\gamma,\rho); \\
\frac{\partial^2 \Omega_2'(\gamma,\rho)}{\partial \rho^2} &= \left(\frac{\gamma}{\rho} - \frac{12}{\gamma\rho^3}\right) \Omega_2(\gamma,\rho) + \left(\frac{6}{\rho^2} - \gamma^2\right) \Omega_2'(\gamma,\rho); \\
\frac{\partial}{\partial \rho}\left[\frac{1}{\rho} \Omega_2(\gamma,\rho)\right] &= -\frac{1}{\rho^2} \Omega_2(\gamma,\rho) + \frac{\gamma}{\rho} \Omega_2'(\gamma,\rho); \\
\frac{\partial^2}{\partial \rho^2}\left[\frac{1}{\rho} \Omega_2(\gamma,\rho)\right] &= \left(\frac{6}{\rho^3} - \frac{\gamma^2}{\rho}\right) \Omega_2(\gamma,\rho) - \frac{3\gamma}{\rho^2} \Omega_2'(\gamma,\rho); \\
\frac{\partial}{\partial \rho}[\rho \Omega_2'(\gamma,\rho)] &= \left(\frac{4}{\gamma\rho} - \gamma\rho\right) \Omega_2(\gamma,\rho).
\end{aligned}\right\} \quad (A\text{-}11)$$

$$\left.\begin{aligned}
\frac{\partial G(\gamma,\rho)}{\partial \rho} &= \frac{1}{\rho} K(\gamma)\, G'(\gamma,\rho); \\
K(\gamma)\, \frac{\partial G'(\gamma,\rho)}{\partial \rho} &= \left(\gamma^2 \rho + \frac{4}{\rho}\right) G(\gamma,\rho).
\end{aligned}\right\} \quad (A\text{-}12)$$

4. Particular Values

$$\left.\begin{aligned}
&\lim_{x\to 0}\left[\frac{1}{x^2} J_2(x)\right] = \frac{1}{8}; \qquad \lim_{x\to 0}\left[\frac{1}{x} J_2'(x)\right] = \frac{1}{4}; \\
&\lim_{x\to 0}\left[x^2 Y_2(x)\right] = -\frac{4}{\pi}; \qquad \lim_{x\to 0}\left[x^3 Y_2'(x)\right] = \frac{8}{\pi}; \\
&\lim_{x\to 0}\left[x^2 K_2(x)\right] = 2; \qquad \lim_{x\to 0}\left[x^3 K_2'(x)\right] = -4.
\end{aligned}\right\} \quad (A\text{-}13)$$

$$K(0) \equiv \lim_{x\to 0} K(x) = -2. \quad (A\text{-}14)$$

$$\lim_{x\to 0}\left[x^6 Y(x)\right] = \frac{64}{\pi^2}. \quad (A\text{-}15)$$

$$\left.\begin{aligned}
&\Omega_2(\gamma, 1) = \frac{2}{\pi\gamma};^* \\
&\Omega_2'(\gamma, 1) = 0; \\
&\lim_{\gamma\to 0}\left[\gamma\Omega_2(\gamma,\rho)\right] = \frac{1}{\pi}\left(\rho^2 + \frac{1}{\rho^2}\right); \\
&\lim_{\gamma\to 0}\left[\gamma^2\Omega_2'(\gamma,\rho)\right] = \frac{2}{\pi}\left(\rho - \frac{1}{\rho^3}\right).
\end{aligned}\right\} \quad \text{(A-16)}$$

$$\left.\begin{aligned}
&G(\gamma,1) = 1; \\
&G'(\gamma,1) = 1; \\
&G(0,\rho) \equiv \lim_{\gamma\to 0} G(\gamma,\rho) = \frac{1}{\rho^2}; \\
&G'(0,\rho) \equiv \lim_{\gamma\to 0} G'(\gamma,\rho) = \frac{1}{\rho^2}.
\end{aligned}\right\} \quad \text{(A-17)}$$

$$\left.\begin{aligned}
&f_1(0) = 12\alpha; \\
&f_2(0) = 6; \\
&f_3(0) = 12; \\
&f_4(0) = 12(1-\alpha); \\
&\lim_{\gamma\to 0}\left[\frac{\Delta(\gamma)}{\gamma^2}\right] = -12.
\end{aligned}\right\} \quad \text{(A-18)}$$

*See Ref. 15, p. 79, No. 28.

Appendix B. Asymptotic Expansions

In this Appendix are listed the asymptotic expansions, valid for large values of the argument, which are needed for the numerical computations and accompanying analysis. In order to indicate the numerical accuracy of the expansion for a function, say f(x), the notation

$$f(10) = 3.141593 \sim 3.141576 \tag{B-1}$$

will be used to indicate that f(10) found by substituting tabulated values into the definition of f is 3.141593, while the listed expansion gives 3.141576 for $x = 10$. In general, the value of x for which the comparison is made is the smallest for which the expansion is used in the numerical computations.

1. Bessel Functions $K_2(x)$, $K_2'(x)$, $J_2(x)$, $J_2'(x)$, $Y_2(x)$, and $Y_2'(x)$

From Ref. 17, p. XXXIV, we have, after some rearrangement,

$$\left.\begin{aligned} K_2(x) &\sim \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \sum_{m=0} \frac{k_m}{x^m}; \\ K_2'(x) &\sim -\left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \sum_{m=0} \frac{k_m'}{x^m}. \end{aligned}\right\} \tag{B-2}$$

$$\left.\begin{aligned} J_2(x) &\sim \left(\frac{2}{\pi x}\right)^{1/2} \sum_{m=0} \frac{k_m}{x^m} \cos \beta_m(x); \\ J_2'(x) &\sim -\left(\frac{2}{\pi x}\right)^{1/2} \sum_{m=0} \frac{k_m'}{x^m} \sin \beta_m(x); \\ Y_2(x) &\sim \left(\frac{2}{\pi x}\right)^{1/2} \sum_{m=0} \frac{k_m}{x^m} \sin \beta_m(x); \\ Y_2'(x) &\sim \left(\frac{2}{\pi x}\right)^{1/2} \sum_{m=0} \frac{k_m'}{x^m} \cos \beta_m(x), \end{aligned}\right\} \tag{B-3}$$

with

$$\beta_m(x) \equiv x + \frac{m\pi}{2} - \frac{5\pi}{4}, \tag{B-4}$$

and

$$k_0 \equiv 1;$$

$$k_m \equiv \frac{(16-1^2)(16-3^2)\cdots[16-(2m-1)^2]}{m!\,2^{3m}}, \qquad m = 1,2,3,\ldots;$$

$$k'_0 \equiv 1;$$

$$k'_m \equiv \frac{(16-1^2)(16-3^2)\cdots[16-(2m-3)^2](15+4m^2)}{m!\,2^{3m}}, \qquad m = 1,2,3,\ldots . \tag{B-5}$$

It can be easily seen from Eqs. (B-5) that

$$k_m = \frac{(5-2m)(3+2m)}{8m} k_{m-1};$$

$$k'_m = \frac{15+4m^2}{8m} k_{m-1}, \qquad m = 1,2,3,\ldots . \tag{B-6}$$

Using Eqs. (B-6), we have

$$k_1 = 2^{-3}(15) = 1.875;$$
$$k_2 = 2^{-7}(105) = 0.8203125;$$
$$k_3 = -2^{-10}(315) = -0.30761719;$$
$$k_4 = 2^{-15}(10{,}395) = 0.31723022;$$
$$k_5 = -2^{-18}(135{,}135) = -0.51549911;$$
$$k_6 = 2^{-22}(4{,}729{,}725) = 1.1276543;$$
$$k_7 = -2^{-25}(103{,}378{,}275) = -3.0809127;$$
$$k'_1 = 2^{-3}(19) = 2.375;$$
$$k'_2 = 2^{-7}(465) = 3.6328125;$$
$$k'_3 = 2^{-10}(1785) = 1.7431641;$$
$$k'_4 = -2^{-15}(24{,}885) = -0.75942993;$$
$$k'_5 = 2^{-18}(239{,}085) = 0.91203690;$$
$$k'_6 = -2^{-22}(7{,}162{,}155) = -1.7075908;$$
$$k'_7 = 2^{-25}(142{,}567{,}425) = 4.2488404. \tag{B-7}$$

By Eqs. (B-2) and (B-7), and Ref. 17,

$$e^{10}\,K_2(10) = 0.47378525 \sim 0.47378522;$$

$$e^{10}\,K_2'(10) = -0.50552362 \sim -0.50552366. \tag{B-8}$$

2. Y(x)

From Ref. 17, p. XXXIV, we have

$$Y(x) \equiv [J_2'(x)]^2 + [Y_2'(x)]^2 \sim \frac{2}{\pi x}\left[1 + \sum_{m=1} \frac{y_{2m}}{x^{2m}}\right], \tag{B-9}$$

with

$$y_{2m} \equiv -\frac{1 \cdot 1 \cdot 3 \cdots (2m-3)}{m!}\;\frac{(16-1^2)(16-3^2)\cdots[16-(2m-3)^2][16-(2m+1)(2m-1)^2]}{2^{3m}}. \tag{B-10}$$

Therefore,

$$y_2 = -2^{-3}(13) = -1.625;$$

$$y_4 = 2^{-7}(435) = 3.3984375;$$

$$y_6 = 2^{-10}(16{,}695) = 16.303711;$$

$$y_8 = -2^{-15}(2{,}008{,}125) = -61.283112, \tag{B-11}$$

and

$$Y(10) = 0.062650098 \sim 0.062650104. \tag{B-12}$$

3. $\Omega_2(\eta, \rho)$

Define

$$R \equiv \rho - 1. \tag{B-13}$$

Substituting Eqs. (B-3) into the definition (A-3) of $\Omega_2(\eta, \rho)$ and grouping together like powers of η, we arrive at

$$\Omega_2(\eta, \rho) \equiv Y_2'(\eta)\,J_2(\eta, \rho) - J_2'(\eta)\,Y_2(\eta, \rho) \sim \frac{2}{\pi\eta\sqrt{\rho}} \sum_{m=0} \frac{\omega_m(R)}{(\eta\rho)^m} \cos\left(\eta R - \frac{m\pi}{2}\right), \tag{B-14}$$

with

$$\omega_m(R) = \sum_{j=0}^{m} (-1)^j (1+R)^{m-j} k_j k'_{m-j}. \tag{B-15}$$

The result of putting k_m and k'_m from Eqs. (B-7) into Eq. (B-15) is

$$\begin{aligned}
\omega_0(R) &= 1; \\
\omega_1(R) &= 2^{-3}(4+19R); \\
\omega_2(R) &= 2^{-7}(15R)(24+31R); \\
\omega_3(R) &= 2^{-10}(15)(-192 - 440R - 108R^2 + 119R^3); \\
\omega_4(R) &= -2^{-15}(315R)(640 + 1184R + 656R^2 + 79R^3); \\
\omega_5(R) &= 2^{-18}(315)(5120 + 14{,}592R + 19{,}200R^2 + 13{,}520R^3 \\
&\quad + 4980R^4 + 759R^5); \\
\omega_6(R) &= 2^{-22}(945R)(71{,}680 + 193{,}920R + 236{,}160R^2 + 154{,}400R^3 \\
&\quad + 53{,}064R^4 + 7579R^5); \\
\omega_7(R) &= 2^{-25}(4725)(114{,}688 + 424{,}960R + 1{,}042{,}944R^2 + 1{,}561{,}216R^3 \\
&\quad + 1{,}422{,}016R^4 + 775{,}368R^5 + 233{,}948R^6 + 30{,}173R^7).
\end{aligned} \tag{B-16}$$

For example,

$$\Omega_2(10, 1.2) = -0.019640844 \sim -0.019640851. \tag{B-17}$$

4. <u>$G(\gamma, \rho)$ and $G'(\gamma, \rho)$</u>

The asymptotic expansions in γ for $G(\gamma, \rho)$ and $G'(\gamma, \rho)$, defined by Eqs. (A-5), can be found from Eqs. (B-2) by dividing the expansions for $K_2(\gamma\rho)$ and $K_2'(\gamma\rho)$ by those for $K_2(\gamma)$ and $K_2'(\gamma)$, respectively. In general, if

$$1 + \sum_{m=1} \alpha_m X^{-m} = \frac{1 + \sum_{m=1} \beta_m X^{-m}}{1 + \sum_{m=1} \gamma_m X^{-m}}, \tag{B-18}$$

where the β_m and γ_m are known and the α_m are to be found, then it is easily verified that

$$\alpha_1 = \beta_1 - \gamma_1;$$

$$\alpha_m = \beta_m - \gamma_m - \sum_{j=1}^{m-1} \gamma_j \alpha_{m-j}, \quad m = 2,3,4, \ldots . \tag{B-19}$$

In particular, if

$$\left.\begin{aligned} \beta_m &= \gamma_m \rho^{-m}, \\ &\text{then} \\ \alpha_1 &= -\gamma_1 R_1; \\ \alpha_m &= -\gamma_m R_m - \sum_{j=1}^{m-1} \gamma_j \alpha_{m-j}, \end{aligned}\right\} \tag{B-20}$$

where

$$R_m \equiv (\rho^m - 1)/\rho^m. \tag{B-21}$$

If we let

$$G(\gamma, \rho) \sim \rho^{-1/2}\, e^{-\gamma R}\left[1 + \sum_{m=1}^{7} g_m(\rho)\, \gamma^{-m}\right];$$

$$G'(\gamma, \rho) \sim \rho^{1/2}\, e^{-\gamma R}\left[1 + \sum_{m=1}^{7} g'_m(\rho)\, \gamma^{-m}\right], \tag{B-22}$$

then by Eqs. (A-5), (B-2), (B-7), (B-18), (B-20), and (B-22),

$$\begin{aligned} g_1 &= -2^{-3}(15)\, R_1; \\ g_2 &= 2^{-7}(15)(30R_1 - 7R_2); \\ g_3 &= 2^{-10}(45)(-115R_1 + 35R_2 + 7R_3); \\ g_4 &= 2^{-15}(45)(4380R_1 - 1610R_2 - 420R_3 - 231R_4); \\ g_5 &= 2^{-18}(135)(-10595R_1 + 5110R_2 + 1610R_3 + 1155R_4 + 1001R_5); \\ g_6 &= 2^{-22}(675)(10314R_1 - 14833R_2 - 6132R_3 - 5313R_4 - 6006R_5 - 7007R_6); \\ g_7 &= 2^{-25}(2025)(144{,}885R_1 + 12{,}033R_2 + 14{,}833R_3 + 16{,}863R_4 \\ &\quad + 23{,}023R_5 + 35{,}035R_6 + 51{,}051R_7), \end{aligned} \tag{B-23}$$

and

$$g_1' = -2^{-3}(19)\,R_1;$$
$$g_2' = 2^{-7}(722R_1 - 465R_2);$$
$$g_3' = 2^{-10}(-4883R_1 + 8835R_2 - 1785R_3);$$
$$g_4' = 2^{-15}(-164{,}692R_1 - 239{,}010R_2 + 135{,}660R_3 + 24{,}885R_4);$$
$$g_5' = 2^{-18}(4{,}619{,}983R_1 - 2{,}015{,}310R_2 - 917{,}490R_3 - 472{,}815R_4 - 239{,}085R_5);$$
$$g_6' = 2^{-22}(-37{,}060{,}754R_1 + 113{,}068{,}005R_2 - 15{,}472{,}380R_3 + 6{,}395{,}445R_4 + 9{,}085{,}230R_5 + 7{,}162{,}155R_6);$$
$$g_7' = 2^{-25}(-1{,}580{,}376{,}319R_1 - 453{,}506{,}595R_2 + 434{,}035{,}245R_3 + 53{,}925{,}795R_4 - 61{,}444{,}845R_5 - 136{,}080{,}945R_6 - 142{,}567{,}425R_7). \tag{B-24}$$

By the above and Ref. 17,

$$e^{10}\,G(10,2) = 0.64815743 \sim 0.64815763;$$
$$e^{10}\,G'(10,2) = 1.2507200 \sim 1.2507196. \tag{B-25}$$

5. K(γ) and Related Functions

In Ref. 24, pp. 6 and 10, the asymptotic expansion for $K(\gamma)$ is found to be

$$K(\gamma) \sim -\gamma - \tfrac{1}{2} + \sum_{m=1} b_m \gamma^{-m}, \tag{B-26}$$

with

$$-b_1 = b_2 = 2^{-3}(15);$$
$$b_m = \tfrac{1}{2}(-1)^m[m b_{m-1} - (b_{m-2}b_1 + b_{m-3}b_2 + \ldots + b_1 b_{m-2})],$$
$$m = 3,4,5,\ldots\,. \tag{B-27}$$

Therefore, we have

$$K(\gamma) \sim -\gamma - 2^{-1} - 2^{-3}(15)\,\gamma^{-1} + 2^{-3}(15)\,\gamma^{-2} - 2^{-7}(135)\,\gamma^{-3} - 2^{-5}(45)\,\gamma^{-4} + 2^{-10}(7425)\,\gamma^{-5} - 2^{-5}(675)\,\gamma^{-6} + 2^{-15}(1{,}905{,}525)\,\gamma^{-7}. \tag{B-28}$$

By the appropriate multiplication of series, we obtain

$$K^2(\gamma) \sim \gamma^2 + \gamma + 4 - 2^{-3}(15)\,\gamma^{-1} + 2^{-2}(15)\,\gamma^{-2} - 2^{-7}(405)\,\gamma^{-3}$$
$$- 2^{-3}(45)\,\gamma^{-4} + 2^{-10}(37{,}125)\,\gamma^{-5} - 2^{-4}(2025)\,\gamma^{-6}$$

and

$$K^3(\gamma) \sim -\gamma^{-3} - 2^{-1}(3)\,\gamma^2 - 2^{-3}(51)\,\gamma - 2^{-3} - 2^{-7}(1215)\,\gamma^{-1} + 2^{-5}(315)\,\gamma^{-2}$$
$$- 2^{-10}(1755)\,\gamma^{-3} - 2^{-5}(1215)\,\gamma^{-4} + 2^{-15}(5{,}825{,}925)\,\gamma^{-5}. \qquad \text{(B-29)}$$

From the above relations and tabular values, it is found that

$$\begin{aligned} K(10) &= -10.6698894 \sim -10.6698881; \\ K^2(10) &= 113.846540 \sim 113.846509; \\ K^3(10) &= -1214.72999 \sim -1214.72988. \end{aligned} \qquad \text{(B-30)}$$

6. <u>$f_1(\gamma)$, $f_2(\gamma)$, $f_3(\gamma)$, and $f_4(\gamma)$</u>

By Eqs. (A-6), (B-28), and (B-29),

$$f_1(\gamma) \sim -\gamma^3 + 3\alpha\gamma^2 + 2^{-3}(24\alpha + 15)\,\gamma + 2^{-2}(48\alpha - 15) + 2^{-7}(-720\alpha + 405)\,\gamma^{-1}$$
$$+ 2^{-3}(90\alpha + 45)\,\gamma^{-2} - 2^{-10}(9720\alpha + 37{,}125)\,\gamma^{-3} + 2^{-4}(-270\alpha + 2025)\,\gamma^{-4};$$

$$f_2(\gamma) \sim \gamma^3 + 2^{-1}(3)\,\gamma^2 + 2^{-3}(39)\,\gamma - 2^{-3}(3) + 2^{-7}(855)\,\gamma^{-1} - 2^{-5}(135)\,\gamma^{-2}$$
$$- 2^{-10}(4185)\,\gamma^{-3} + 2^{-4}(405)\,\gamma^{-4} - 2^{-15}(2{,}618{,}325)\,\gamma^{-5};$$

$$f_3(\gamma) \sim \gamma^4 + 2\gamma^3 + 9\gamma^2 + 2^{-2}(15)\,\gamma + 2^{-2}(33) + 2^{-6}(135)\,\gamma^{-1} + 2^{-4}(45)\,\gamma^{-2}$$
$$- 2^{-9}(7425)\,\gamma^{-3} + 2^{-4}(675)\,\gamma^{-4} - 2^{-14}(1{,}905{,}525)\,\gamma^{-5};$$

$$f_4(\gamma) \sim \gamma^4 + (-\alpha + 1)\,\gamma^3 + 2^{-1}(-3\alpha + 15)\,\gamma^2 + 2^{-3}(-63\alpha + 15)\,\gamma + 2^{-3}(-9\alpha + 81)$$
$$+ 2^{-7}(-1575\alpha + 135)\,\gamma^{-1} + 2^{-5}(315\alpha + 45)\,\gamma^{-2} + 2^{-10}(945\alpha - 7425)\,\gamma^{-3}$$
$$+ 2^{-5}(-945\alpha + 675)\,\gamma^{-4} + 2^{-15}(3{,}331{,}125\alpha - 1{,}905{,}525)\,\gamma^{-5}, \qquad \text{(B-31)}$$

and

$$\begin{aligned} f_1(10) &= -984.6540 + 341.5396\alpha \sim -984.6509 + 341.5388\alpha; \\ f_2(10) &= 1198.99861 \sim 1198.99843; \\ f_3(10) &= 12945.97788 \sim 12945.97763; \\ f_4(10) &= 11778.98894 - 1231.00828\alpha \sim 11778.98881 - 1231.00805\alpha. \end{aligned} \qquad \text{(B-32)}$$

7. $\Delta(\gamma)$ and $-\gamma^5/\Delta(\gamma)$

From Eqs. (A-6), (B-28), (B-29), and (B-31), we have

$$\Delta(\gamma) \sim \gamma^5[-1-(\alpha+2)\gamma^{-1} - 2^{-3}(24\alpha+57)\gamma^{-2} - 2^{-2}(12\alpha+15)\gamma^{-3}$$
$$+ 2^{-7}(144\alpha+549)\gamma^{-4} - 2^{-3}(135)\gamma^{-5} + 2^{-10}(6840\alpha+2115)\gamma^{-6}$$
$$+ 2^{-4}(-315\alpha+1665)\gamma^{-7}]. \quad \text{(B-33)}$$

Application of Eqs. (B-18) and (B-19) to Eq. (B-33) yields

$$-\frac{\gamma^5}{\Delta(\gamma)} \sim 1 + \sum_{m=1}^{7} \delta_m(\alpha)\gamma^{-m}, \quad \text{(B-34)}$$

with

$$\delta_1 = -(\alpha+2);$$

$$\delta_2 = 2^{-3}(8\alpha^2+8\alpha-25);$$

$$\delta_3 = 2^{-2}(-4\alpha^3+45\alpha+67);$$

$$\delta_4 = 2^{-7}(128\alpha^4-128\alpha^3-2352\alpha^2-3344\alpha+71);$$

$$\delta_5 = 2^{-4}(-16\alpha^5+32\alpha^4+392\alpha^3+340\alpha^2-1274\alpha-2147);$$

$$\delta_6 = 2^{-10}(1024\alpha^6-3072\alpha^5-30{,}336\alpha^4-3200\alpha^3+223{,}320\alpha^2$$
$$+419{,}640\alpha+229{,}399);$$

$$\delta_7 = 2^{-6}(-64\alpha^7+256\alpha^6+2160\alpha^5-1744\alpha^4-25{,}316\alpha^3-39{,}816\alpha^2$$
$$+2411\alpha+46{,}750). \quad \text{(B-35)}$$

Recall from Eq. (12) that

$$\alpha \equiv 2(1-\nu), \quad \text{(B-36)}$$

where ν is Poisson's ratio. Thus Poisson's ratios of 1/2 and 1/4 correspond to values of α of 1 and 3/2, respectively. For $\alpha = 1$, from Eqs. (B-34) and (B-35) we have that

$$-\frac{\gamma^5}{\Delta(\gamma)} \sim 1 - 3\gamma^{-1} - 2^{-3}(9)\gamma^{-2} + 27\gamma^{-3} - 2^{-7}(5625)\gamma^{-4} - 2^{-4}(2673)\gamma^{-5}$$
$$+ 2^{-10}(836{,}775)\gamma^{-6} - 2^{-6}(15{,}363)\gamma^{-7}, \quad \text{(B-37)}$$

whereas for $\alpha = 3/2$

$$-\frac{\gamma^5}{\Delta(\gamma)} \sim 1 - 2^{-1}(7)\,\gamma^{-1} + 2^{-3}(5)\,\gamma^{-2} + 2^{-2}(121)\,\gamma^{-3} - 2^{-7}(10{,}021)\,\gamma^{-4}$$

$$- 2^{-5}(3859)\,\gamma^{-5} + 2^{-10}(1{,}185{,}289)\,\gamma^{-6} - 2^{-6}(115{,}265)\,\gamma^{-7}. \qquad \text{(B-38)}$$

By tabular values and the above expansions we have

$$\Delta(10) = -127{,}473.09 - 13{,}288.263\alpha \sim -127{,}472.74 - 13{,}288.279\alpha; \qquad \text{(B-39)}$$

for $\alpha = 1$

$$\left.\begin{aligned} &-10^5/\Delta(10) = 0.710422 \sim 0.710478 \\ &\text{and for } \alpha = 3/2 \\ &-10^5/\Delta(10) = 0.678401 \sim 0.678443. \end{aligned}\right\} \qquad \text{(B-40)}$$

8. $F_1(\gamma,\alpha)$, $F_2(\gamma,\alpha)$, and $F_3(\gamma,\alpha)$

By definition [see Eqs. (74)],

$$F_1(\gamma,\alpha) \equiv -\frac{\alpha\gamma^2}{\Delta(\gamma)}\, f_1(\gamma);$$

$$F_2(\gamma,\alpha) \equiv -\frac{2\alpha\gamma^2}{\Delta(\gamma)}\, K(\gamma)\, f_2(\gamma);$$

$$F_3(\gamma,\alpha) \equiv -\frac{\gamma^2}{\Delta(\gamma)}\, K^2(\gamma)\, f_3(\gamma). \qquad \text{(B-41)}$$

The expansion for $f_1(\gamma)$ is given in Eqs. (B-31); by Eqs. (61) we have

$$K(\gamma)\, f_2(\gamma) = \gamma^2 K(\gamma) - (\gamma^2 + 3)\, K^2(\gamma);$$

$$K^2(\gamma)\, f_3(\gamma) = (\gamma^2 + 2)(\gamma^2 + 6)\, K^2(\gamma) - 2\gamma^2 K^3(\gamma), \qquad \text{(B-42)}$$

so that, using Eqs. (B-28) and (B-29),

$$K(\gamma)\, f_2(\gamma) \sim -\gamma^4 - 2\gamma^3 - 2^{-1}(15)\,\gamma^2 - 3\gamma - 2^{-3}(111) + 2^{-6}(495)\,\gamma^{-1}$$

$$- 2^{-5}(225)\,\gamma^{-2} - 2^{-8}(4995)\,\gamma^{-3} + 2^{-5}(3915)\,\gamma^{-4};$$

$$K^2(\gamma)\, f_3(\gamma) \sim \gamma^6 + 3\gamma^5 + 15\gamma^4 + 2^{-3}(151)\,\gamma^3 + 48\gamma^2 + 2^{-7}(1641)\,\gamma + 2^{-4}(843)$$

$$- 2^{-10}(8325)\,\gamma^{-1} - 2^{-3}(405)\,\gamma^{-2}. \qquad \text{(B-43)}$$

By tabular values and Eqs. (B-43)

$$\left.\begin{aligned} K(10)\, f_2(10) &= -12{,}793.1825 \sim -12{,}793.1792 \\ \text{and} \qquad & \\ K^2(10)\, f_3(10) &= 1{,}473{,}854.79 \sim 1{,}473{,}854.57. \end{aligned}\right\} \quad \text{(B-44)}$$

The leading terms in the expansions for F_1, F_2, and F_3 are found from the substitution of Eqs. (B-31), (B-43), (B-34), and (B-35) into (B-41). They are given by

$$F_1(\gamma, \alpha) \sim -\alpha + 2\alpha(2\alpha + 1)\,\gamma^{-1} + \alpha(-4\alpha^2 - 4\alpha + 5)\,\gamma^{-2};$$

$$F_2(\gamma, \alpha) \sim -2\alpha\gamma + 2\alpha^2 + 2^{-2}\alpha(-8\alpha^2 + 8\alpha - 3)\,\gamma^{-1};$$

$$F_3(\gamma, \alpha) \sim \gamma^3 + (-\alpha + 1)\,\gamma^2 + 2^{-3}(8\alpha^2 - 16\alpha + 47)\,\gamma + 2^{-2}(-4\alpha^3 + 12\alpha^2 - 3\alpha - 15)$$
$$+ 2^{-7}(128\alpha^4 - 512\alpha^3 - 432\alpha^2 + 480\alpha + 1815)\,\gamma^{-1}. \qquad \text{(B-45)}$$

9. <u>$\tilde{F}_1(\gamma, \alpha)$, $\tilde{F}_2(\gamma, \alpha)$, and $\tilde{F}_3(\gamma, \alpha)$</u>

From Eqs. (74), (75), and (76), (B-31), (B-43), (B-37), and (B-38), we find that

$$\tilde{F}_j(\gamma, \alpha) \sim \sum_{m=1}^{N_j} b_{jm}(\alpha)\,\gamma^{-m}, \quad j = 1,2,3, \qquad \text{(B-46)}$$

where the $b_{jm}(\alpha)$ are given in Table B-I for Poisson's ratios of 1/2 and 1/4 (α = 1 and 3/2). We have

$$\tilde{F}_1(10, 1) = 0.543117 \sim 0.543155;$$

$$\tilde{F}_2(10, 1) = -0.17712 \sim -0.17771;$$

$$\tilde{F}_3(10, 1) = 0.8093 \sim 0.8181;$$

$$\tilde{F}_1(10, 3/2) = 1.019342 \sim 1.019428;$$

$$\tilde{F}_2(10, 3/2) = -0.53672 \sim -0.53764;$$

$$\tilde{F}_3(10, 3/2) = 0.1143 \sim 0.1218. \qquad \text{(B-47)}$$

TABLE B-I. The Coefficients b_{jm} in the Asymptotic Expansions for $\tilde{F}_1$, $\tilde{F}_2$, and $\tilde{F}_3$

	$\nu = 1/2, \alpha = 1$			$\nu = 1/4, \alpha = 3/2$		
m	b_{1m}	b_{2m}	b_{3m}	b_{1m}	b_{2m}	b_{3m}
1	6	-3/4	1479/128	12	-27/8	483/128
2	-3	-21/2	-741/16	-15	-99/4	-1443/32
3	-147/4	-831/64	129,687/1024	-213/4	3735/128	235,701/1024
4	369/4	1683/8	-549/64	1959/8	10,053/32	-2781/8
5	11,637/64	-225,051/512		33/2	-1,421,343/1024	
6	-22,455/16	-12,519/16		-22,809/8	20,691/32	
7	58,141/512			38,081/512		
8	38,907/4			70,521/8		

10. $F_4(\gamma, \rho, \alpha)$ and $F_5(\gamma, \rho, \alpha)$

The functions F_4 and F_5 are defined in Eqs. (99) as

$$F_4(\gamma, \rho, \alpha) \equiv - \frac{2\alpha\gamma^2}{\Delta(\gamma)}\{[(\alpha+2)\, f_2(\gamma)+f_4(\gamma)]\; G(\gamma, \rho)+K(\gamma)\, f_2(\gamma)\; G'(\gamma, \rho)\};$$

$$F_5(\gamma, \rho, \alpha) \equiv - \frac{\gamma^2}{\Delta(\gamma)}\Big[\{[2K(\gamma) - \gamma^2 - 4]\; f_3(\gamma) + 4\alpha f_2(\gamma)\}\; G(\gamma, \rho) + K^2(\gamma)\; f_3(\gamma)\; G'(\gamma, \rho)\Big]. \tag{B-48}$$

The expansions for $K(\gamma)\, f_2(\gamma)$ and $K^2(\gamma)\, f_3(\gamma)$ are given by Eqs. (B-43). By means of Eqs. (B-31), (B-28), and (B-29), it can be shown that

$$\begin{aligned}(\alpha+2)\, f_2(\gamma) + f_4(\gamma) \sim{}& \gamma^4 + 3\gamma^3 + 2^{-1}(21)\,\gamma^2 + 2^{-3}(-24\alpha+93)\,\gamma + 2^{-3}(-12\alpha+75)\\ &+ 2^{-7}(-720\alpha+1845)\,\gamma^{-1} + 2^{-5}(180\alpha - 225)\,\gamma^{-2}\\ &+ 2^{-10}(-3240\alpha - 15{,}795)\,\gamma^{-3} + 2^{-5}(-135\alpha+2295)\,\gamma^{-4}\\ &+ 2^{-15}(712{,}800\alpha - 7{,}142{,}175)\,\gamma^{-5};\end{aligned}$$

$$\begin{aligned}[2K(\gamma) - \gamma^2 - 4]\; f_3(\gamma) + 4\alpha f_2(\gamma) ={}& -(\gamma^2+2)(\gamma^2+4)(\gamma^2+6) + 4(\gamma^4+6\gamma^2+6)\; K(\gamma)\\ &- 4\gamma^2K^2(\gamma) + 4\alpha f_2(\gamma) \sim -\gamma^6 - 4\gamma^5 - 18\gamma^4\\ &+ 2^{-1}(8\alpha - 71)\,\gamma^3 + 2^{-1}(12\alpha - 129)\,\gamma^2\\ &+ 2^{-5}(624\alpha - 2103)\,\gamma + 2^{-3}(-12\alpha - 285)\\ &+ 2^{-8}(6840\alpha - 7335)\,\gamma^{-1} + 2^{-3}(-135\alpha - 405)\,\gamma^{-2}\\ &+ 2^{-13}(-133{,}920\alpha + 1{,}935{,}765)\,\gamma^{-3}.\end{aligned} \tag{B-49}$$

By tabulated values and Eqs. (B-49), we have, for $\alpha = 1$,

$$3f_2(10) + f_4(10) = 14{,}144.97649 \sim 14{,}144.97604;$$

$$[2K(10) - 104]\, f_3(10) + 4f_2(10) = 1{,}617{,}850.008 \sim 1{,}617{,}849.961. \qquad \text{(B-50)}$$

The leading terms in the asymptotic expansions of F_4 and F_5 are found by substituting from Eqs. (B-43), (B-49), (B-34), (B-35), and (B-22) into Eq. (B-48). This results in

$$\begin{aligned}
F_4(\gamma,\rho,\alpha) \sim 2\alpha\rho^{-\frac{1}{2}} e^{-\gamma R}\{&-R\gamma + [1+\alpha R + g_1(\rho) - \rho g_1'(\rho)] \\
&+ [-\alpha + 1 + 2^{-3}(-8\alpha^2 + 8\alpha - 3)\, R + (-\alpha+1)\, g_1(\rho) \\
&+ \alpha\rho g_1'(\rho) + g_2(\rho) - \rho g_2'(\rho)]\, \gamma^{-1}\};
\end{aligned}$$

$$\begin{aligned}
F_5(\gamma,\rho,\alpha) \sim \rho^{-\frac{1}{2}} e^{-\gamma R}\{&R\gamma^3 + [-1 + (-\alpha+1)\, R - g_1(\rho) + \rho g_1'(\rho)]\, \gamma^2 \\
&+ [\alpha - 1 + 2^{-3}(8\alpha^2 - 16\alpha + 47)\, R + (\alpha - 2)\, g_1(\rho) \\
&+ (-\alpha+1)\, \rho g_1'(\rho) - g_2(\rho) + \rho g_2'(\rho)]\, \gamma \\
&+ [2^{-1}(-2\alpha^2 + 12\alpha - 15) + 2^{-2}(-4\alpha^3 + 12\alpha^2 - 3\alpha - 15)\, R \\
&+ 2^{-3}(-8\alpha^2 + 24\alpha - 55)\, g_1(\rho) + 2^{-3}(8\alpha^2 - 16\alpha + 47)\, \rho g_1'(\rho) \\
&+ (\alpha - 2)\, g_2(\rho) + (-\alpha + 1)\, \rho g_2'(\rho) - g_3(\rho) + \rho g_3'(\rho)] \\
&+ [2^{-3}(8\alpha^3 - 56\alpha^2 + 3\alpha + 75) + 2^{-7}(128\alpha^4 - 512\alpha^3 - 432\alpha^2 + 480\alpha + 1815)\, R \\
&+ 2^{-2}(4\alpha^3 - 16\alpha^2 + 27\alpha - 15)\, g_1(\rho) + 2^{-2}(-4\alpha^3 + 12\alpha^2 - 3\alpha - 15)\, \rho g_1'(\rho) \\
&+ 2^{-3}(-8\alpha^2 + 24\alpha - 55)\, g_2(\rho) + 2^{-3}(8\alpha^2 - 16\alpha + 47)\, \rho g_2'(\rho) + (\alpha - 2)\, g_3(\rho) \\
&+ (-\alpha + 1)\, \rho g_3'(\rho) - g_4(\rho) + \rho g_4'(\rho)]\, \gamma^{-1}\},
\end{aligned} \qquad \text{(B-51)}$$

where the $g_i(\rho)$ and $g_i'(\rho)$ are given by Eqs. (B-23) and (B-24), and R by Eq. (B-13).

11. <u>$Q_{mn}(x)$</u>

In Eq. (107), Q_{mn} is defined as

$$Q_{mn}(x) \equiv \frac{1}{m!}\int_0^\infty t^m g^n\!\left(\frac{t}{x}\right) e^{-t}\, dt, \qquad m,n = 0,1,2,3, \ldots . \qquad \text{(B-52)}$$

The asymptotic expansion for $Q_{mn}(x)$ can be obtained by writing [see Eq. (58) and Ref. 20, p. 88, #748]

$$g^n\left(\frac{t}{x}\right) \equiv \left(\frac{x^2}{t^2+x^2}\right)^n$$

$$= 1 - n\left(\frac{t}{x}\right)^2 + \frac{n(n+1)}{2}\left(\frac{t}{x}\right)^4 - \ldots (-1)^k \frac{(n+k-1)!}{(n-1)!k!}\left(\frac{t}{x}\right)^{2k} + \ldots,$$

$$|x| > |t|. \tag{B-53}$$

Substituting Eq. (B-53) into Eq. (B-52) and integrating term by term, using

$$\int_0^\infty t^{m+2k}\, e^{-t}\, dt = (m+2k)! \tag{B-54}$$

from Ref. 20, p. 63, #493, we have

$$Q_{mn}(x) \sim 1 - \frac{n(m+2)!}{m!\, x^2} + \frac{n(n+1)(m+4)!}{2m!\, x^4} \ldots + (-1)^k \frac{(n+k-1)!\,(m+2k)!}{(n-1)!\,k!\,m!\,x^{2k}} + \ldots . \tag{B-55}$$

The same expansion is obtained for the combinations of m and n needed for this problem if the asymptotic expansion for the sine integral and cosine integral from Ref. 22 are used together with Eqs. (110) and (112).

If we write

$$q_{mnk} \equiv \frac{(-1)^k (n+k-1)!\,(m+2k)!}{(n-1)!\,k!\,m!}, \qquad m, k = 0,1,2, \ldots; \qquad n = 1,2,3, \ldots, \tag{B-56}$$

then Eq. (B-55) becomes

$$Q_{mn}(x) \sim \sum_{k=0}^{N_{mn}(x)} q_{mnk}\, x^{-2k}, \qquad m = 0,1,2, \ldots; \qquad n = 1,2,3, \ldots \tag{B-57}$$

where $N_{mn}(x)$ is the highest value of k for which $|q_{mnk}\, x^{-2k}|$ is monotonically decreasing with increasing k. For purposes of numerical computation it is convenient to rewrite Eq. (B-56) as

$$q_{mnk} = -(1/k)(n+k-1)(m+2k)(m+2k-1)\, q_{mn,k-1};$$

$$q_{mn0} = 1; \qquad m = 0,1,2, \ldots; \qquad n, k = 1,2,3, \ldots . \tag{B-58}$$

By Eqs. (B-58), we see that $N_{mn}(x)$ is equal to the largest value of k for which

$$(n+k-1)(m+2k)(m+2k-1)/kx^2 \leq 1. \tag{B-59}$$

It is apparent from the relation (B-59) that the larger that m and n are, the smaller will be the number of valid terms which can be retained in the corresponding asymptotic expansion for a given x. For instance, $N_{32}(11) = 3$, so that $Q_{32}(11)$ would have only four terms (k = 0,1,2,3) in its asymptotic expansion. By making use of Refs. 22 and 25, and Eqs. (110), (112), (B-57), (B-58), and (B-59), we have, for the combinations of m and n which are needed

$$\begin{aligned}
Q_{01}(11) &= 0.9848192 \sim 0.9847521, & N_{01}(11) &= 5;\\
Q_{11}(11) &= 0.956721 \sim 0.955918, & N_{11}(11) &= 5;\\
Q_{02}(11) &= 0.9707702 \sim 0.9703352, & N_{02}(11) &= 5;\\
Q_{12}(11) &= 0.918438 \sim 0.922499, & N_{12}(11) &= 4;\\
Q_{22}(11) &= 0.84997 \sim 0.87222, & N_{22}(11) &= 4;\\
Q_{32}(11) &= 0.77205 \sim 0.70498, & N_{32}(11) &= 3.
\end{aligned} \tag{B-60}$$

In the numerical check of the boundary condition discussed in Part II, $Q_{mn}(x)$ must be computed for a wide range of values of x. The Fortran library subroutine ARCSI1 for the sine integral and cosine integral was used with some modifications in the calculation of $Q_{mn}(x)$ for small x [see Eqs. (110) and (112)]. In ARCSI1 the range of x is broken into three regions: $0 \leq x < 1$, $1 \leq x < 18$, and $18 \leq x$. For $0 \leq x < 1$, power series are used to compute si(x) and Ci(x); for $1 \leq x < 18$, empirically determined ratios of fourth-order polynomials are used; and for $x \geq 18$, asymptotic expansions are employed.

In the program written for this problem, for $0 \leq x < 1$, ARCSI1 was used to compute si(x) and Ci(x). These results were then substituted into Eqs. (110) to find $Q_{01}(x)$ and $Q_{11}(x)$, and finally Q_{01} and Q_{11} were substituted into Eqs. (112) in order to calculate Q_{02}, Q_{12}, Q_{22}, and Q_{32}.

For $x \geq 18$, however, following this same procedure would lead to difficulties. The asymptotic expansion employed in ARCSI1 when substituted into Eqs. (110) yields exactly the same expansion as Eqs. (B-57) for Q_{01} and Q_{11}. Substitution of these values into Eqs. (112) would result in a significant loss in accuracy in the calculated values of Q_{12}, Q_{22}, and Q_{32} since for large x, both Q_{01} and Q_{11} are very close to unity; evaluation of Eqs. (112) for Q_{12}, Q_{22}, and Q_{32} would thus mean multiplying a large number, x^2, times a small number, obtained by subtracting two almost equal quantities, to yield values close to unity but having few significant figures. Therefore, the expansion (B-57) was used for all the combinations of m and n.

In the intermediate range, $1 \le x < 18$, it was found that the use of ARCSI1 to compute si(x) and Ci(x), followed by substitutions into Eqs. (110) and (112), yielded sufficiently accurate results for $1 \le x < 11$, but for $11 \le x < 18$, a serious loss in the number of significant figures in the computed values of Q_{12}, Q_{22}, and Q_{32} was encountered. This was due to the same phenomenon as was discussed above for $x \ge 18$, i.e., the subtraction of almost equal quantities in Eqs. (112) due to the Q_{mn} being close to unity in value. An attempt was made to use the asymptotic expansion (B-57) in the range $11 \le x < 18$, but as can be observed from Eqs. (B-60), this did not give very satisfactory results in this range, particularly for Q_{22} and Q_{32}.

Because of the $(-1)^k$ factor in q_{mnk} defined in Eq. (B-56) the values of the sum in Eq. (B-57) oscillate around the true value of $Q_{mn}(x)$ as successive terms in the sum are added. It was observed that if only half the last term retained in the expansion were added, the resulting sum was significantly closer to the true value of $Q_{mn}(x)$ than if the entire last term were added. Therefore, we shall write, instead of Eq. (B-57),

$$Q_{mn}(x) \sim \sum_{k=0}^{N_{mn}(x)-1} q_{mnk}\, x^{-2k} + \tfrac{1}{2}\, q_{mnk}\, x^{-2k} \Big|_{k=N_{mn}(x)}. \qquad \text{(B-61)}$$

Using in Eq. (B-61) the same q_{mnk} as were used in calculating the asymptotic results given in Eqs. (B-60), we obtain

$$\begin{aligned}
Q_{01}(11) &= 0.9848192 \sim 0.9848220;\\
Q_{11}(11) &= 0.956721 \sim 0.956688;\\
Q_{02}(11) &= 0.9707702 \sim 0.9707549;\\
Q_{12}(11) &= 0.918438 \sim 0.918267;\\
Q_{22}(11) &= 0.84997 \sim 0.85106;\\
Q_{32}(11) &= 0.77205 \sim 0.77326.
\end{aligned} \qquad \text{(B-62)}$$

The asymptotic values given in Eqs. (B-62) are better than those given in Eqs. (B-60). Hence, the asymptotic expansion (B-61) was used in the computer program to approximate the desired $Q_{mn}(x)$ for $x \ge 11$.

12. $F_j(\gamma, 1, \alpha)$, $j = 4,5,6,7,8,9$

By substituting the asymptotic expansions for $K(\gamma)$ and $K^2(\gamma)$ from Eqs. (B-28) and (B-29), the expansions for $f_2(\gamma)$, $f_3(\gamma)$, and $f_4(\gamma)$ from Eqs. (B-31), and the expansion for $\gamma^5/\Delta(\gamma)$ from Eqs. (B-34) and (B-35) into the Eqs. (125) for $F_4(\gamma, 1, \alpha)$ through $F_9(\gamma, 1, \alpha)$, we find the asymptotic expansion for the F_j. The leading terms in these expansions are found to be given by

$$F_4(\gamma, 1, \alpha) \sim 2\alpha[1 + (1 - \alpha)\gamma^{-1}];$$

$$F_5(\gamma, 1, \alpha) \sim -\gamma^2 + (\alpha - 1)\gamma + 2^{-1}(-2\alpha^2 + 12\alpha - 15) + 2^{-3}(8\alpha^3 - 56\alpha^2 + 3\alpha + 75)\gamma^{-1};$$

$$F_6(\gamma, 1, \alpha) \sim 2\alpha - (2\alpha^2 + \alpha)\gamma^{-1};$$

$$F_7(\gamma, 1, \alpha) \sim -\gamma^2 + 2^{-1}(2\alpha - 1)\gamma + 2^{-2}(-4\alpha^2 + 6\alpha - 15)$$
$$+ 2^{-4}(16\alpha^3 - 40\alpha^2 - 30\alpha + 75)\gamma^{-1};$$

$$F_8(\gamma, 1, \alpha) \sim \alpha[\gamma + 2^{-1}(-8\alpha + 5) + 2^{-3}(32\alpha^2 - 16\alpha - 97)\gamma^{-1}];$$

$$F_9(\gamma, 1, \alpha) \sim 2(\alpha - 1)\gamma^2 + (-2\alpha^2 + 3\alpha)\gamma + 2^{-1}(4\alpha^3 - 10\alpha^2 - 3\alpha)$$
$$+ 2^{-3}(-16\alpha^4 + 56\alpha^3 + 86\alpha^2 - 117\alpha)\gamma^{-1}. \qquad \text{(B-63)}$$

The first two of these can also be obtained by letting $\rho = 1$ in Eqs. (B-51), noting that $g_i(1)$ and $g_i'(1)$ are zero, from Eqs. (B-23), (B-24), and (B-21).

13. <u>$\gamma^{-5}\Delta(\gamma)\,\tilde{F}_j(\gamma, 1, \alpha)$, $j = 4,5,6,7,8,9$</u>

By Eqs. (125), (126), and (127) we have

$$\gamma^{-5}\Delta(\gamma)\,\tilde{F}_4(\gamma, 1, \alpha) = -2\alpha\gamma^{-3}\{[\alpha + 2 + K(\gamma)]\,f_2(\gamma) + f_4(\gamma) + \gamma^{-2}\Delta(\gamma)\};$$

$$\gamma^{-5}\Delta(\gamma)\,\tilde{F}_5(\gamma, 1, \alpha) = -4\alpha\gamma^{-3}f_2(\gamma) + \gamma^{-3}[\gamma^2 + 4 - K^2(\gamma) - 2K(\gamma)]\,f_3(\gamma)$$
$$+ [\gamma^{-3} + (-\alpha + 1)\gamma^{-4} + 2^{-1}(2\alpha^2 - 12\alpha + 15)\gamma^{-5}]\,\Delta(\gamma);$$

$$\gamma^{-5}\Delta(\gamma)\,\tilde{F}_6(\gamma, 1, \alpha) = -2\alpha[\gamma^{-3}f_2(\gamma) + \gamma^{-5}\Delta(\gamma)];$$

$$\gamma^{-5}\Delta(\gamma)\,\tilde{F}_7(\gamma, 1, \alpha) = -\gamma^{-3}K(\gamma)\,f_3(\gamma) + [\gamma^{-3} + 2^{-1}(-2\alpha + 1)\gamma^{-4}$$
$$+ 2^{-2}(4\alpha^2 - 6\alpha + 15)\gamma^{-5}]\,\Delta(\gamma);$$

$$\gamma^{-5}\Delta(\gamma)\,\tilde{F}_8(\gamma, 1, \alpha) = \alpha\{-\gamma^{-3}K(\gamma)\,f_1(\gamma) + 4\gamma^{-3}[\alpha + K(\gamma)]\,f_2(\gamma)$$
$$+ 4\gamma^{-3}f_4(\gamma) + [\gamma^{-4} + 2^{-1}(8\alpha - 5)\gamma^{-5}]\,\Delta(\gamma)\};$$

$$\gamma^{-5}\Delta(\gamma)\,\tilde{F}_9(\gamma, 1, \alpha) = 2\alpha\gamma^{-3}[4 - K^2(\gamma)]\,f_2(\gamma) + 2\gamma^{-3}[K^2(\gamma) - \gamma^2 - 4]\,f_3(\gamma)$$
$$+ [2(-\alpha + 1)\gamma^{-3} + (2\alpha^2 - 3\alpha)\gamma^{-4} + 2^{-1}(-4\alpha^3 + 10\alpha^2 + 3\alpha)\gamma^{-5}]\,\Delta(\gamma).$$

(B-64)

Substitution from Eqs. (B-31), (B-33), and (B-43) into Eq. (B-64) yields

$$\gamma^{-5}\Delta(\gamma)\ \widetilde{F}_4(\gamma, 1, \alpha) \sim \alpha[2(\alpha - 1)\ \gamma^{-1} + (12\alpha - 3)\ \gamma^{-2} + 2^{-1}(18\alpha + 33)\ \gamma^{-3}$$

$$+ 2^{-3}(72\alpha - 423)\ \gamma^{-4} + 2^{-3}(-90\alpha + 495)\ \gamma^{-5}$$

$$+ 2^{-7}(-900\alpha + 8415)\ \gamma^{-6} + 2^{-4}(765\alpha - 9540)\ \gamma^{-7}];$$

$$\gamma^{-5}\Delta(\gamma)\ \widetilde{F}_5(\gamma, 1, \alpha) \sim 2^{-3}(-8\alpha^3 + 56\alpha^2 - 3\alpha - 75)\ \gamma^{-1} + 2^{-3}(-24\alpha^3 + 111\alpha^2 + 21\alpha)\ \gamma^{-2}$$

$$+ 2^{-7}(-384\alpha^3 + 1680\alpha^2 - 213\alpha - 7395)\ \gamma^{-3}$$

$$+ 2^{-7}(144\alpha^3 - 315\alpha^2 - 2619\alpha + 6930)\ \gamma^{-4}$$

$$+ 2^{-10}(-24{,}120\alpha^2 + 105{,}525\alpha + 82{,}755)\ \gamma^{-5};$$

$$\gamma^{-5}\Delta(\gamma)\ \widetilde{F}_6(\gamma, 1, \alpha) \sim \alpha[(2\alpha + 1)\ \gamma^{-1} + 2^{-1}(12\alpha + 9)\ \gamma^{-2} + 2^{-2}(24\alpha + 33)\ \gamma^{-3}$$

$$+ 2^{-4}(-36\alpha - 351)\ \gamma^{-4} + 2^{-4}(675)\ \gamma^{-5} + 2^{-8}(-3420\alpha + 1035)\ \gamma^{-6}$$

$$+ 2^{-3}(315\alpha - 2070)\ \gamma^{-7}];$$

$$\gamma^{-5}\Delta(\gamma)\ \widetilde{F}_7(\gamma, 1, \alpha) \sim 2^{-4}(-16\alpha^3 + 40\alpha^2 + 30\alpha - 75)\ \gamma^{-1} + 2^{-4}(-48\alpha^3 + 6\alpha^2 + 45\alpha)\ \gamma^{-2}$$

$$+ 2^{-8}(-768\alpha^3 - 96\alpha^2 - 2394\alpha - 7395)\ \gamma^{-3}$$

$$+ 2^{-8}(288\alpha^3 + 666\alpha^2 + 5463\alpha - 6930)\ \gamma^{-4}$$

$$+ 2^{-11}(-48{,}240\alpha^2 + 14{,}130\alpha + 82{,}755)\ \gamma^{-5};$$

$$\gamma^{-5}\Delta(\gamma)\ \widetilde{F}_8(\gamma, 1, \alpha) \sim \alpha[2^{-3}(-32\alpha^2 + 16\alpha + 97)\ \gamma^{-1} + 2^{-3}(-96\alpha^2 - 87\alpha + 129)\ \gamma^{-2}$$

$$+ 2^{-7}(-1536\alpha^2 - 1824\alpha - 789)\ \gamma^{-3} + 2^{-7}(576\alpha^2 + 2601\alpha + 4680)\ \gamma^{-4}$$

$$+ 2^{-10}(-83{,}160\alpha + 6525)\ \gamma^{-5}$$

$$+ 2^{-10}(27{,}360\alpha^2 + 3825\alpha - 128{,}655)\ \gamma^{-6}];$$

$$\gamma^{-5}\Delta(\gamma)\ \widetilde{F}_9(\gamma, 1, \alpha) \sim \alpha[2^{-3}(16\alpha^3 - 56\alpha^2 - 86\alpha + 117)\ \gamma^{-1}$$

$$+ 2^{-3}(48\alpha^3 - 54\alpha^2 - 327\alpha - 162)\ \gamma^{-2}$$

$$+ 2^{-7}(768\alpha^3 - 672\alpha^2 - 2310\alpha + 3033)\ \gamma^{-3}$$

$$+ 2^{-7}(-288\alpha^3 - 378\alpha^2 - 3069\alpha + 8406)\ \gamma^{-4}$$

$$+ 2^{-10}(48{,}240\alpha^2 - 62{,}370\alpha - 326{,}025)\ \gamma^{-5}]. \qquad \text{(B-65)}$$

Using tabulated values, Eqs. (B-64) and Eqs. (B-65), we find that

$$
\begin{aligned}
10^{-5}\Delta(10)\, \tilde{F}_4(10,1,1) &= 0.111639 \sim 0.111623;\\
10^{-5}\Delta(10)\, \tilde{F}_4(10,1,1.5) &= 0.414797 \sim 0.414773;\\
10^{-5}\Delta(10)\, \tilde{F}_5(10,1,1) &= -0.28516 \sim -0.28448;\\
10^{-5}\Delta(10)\, \tilde{F}_5(10,1,1.5) &= 0.45648 \sim 0.45719;\\
10^{-5}\Delta(10)\, \tilde{F}_6(10,1,1) &= 0.4172298 \sim 0.4172219;\\
10^{-5}\Delta(10)\, \tilde{F}_6(10,1,1.5) &= 0.825169 \sim 0.825157;\\
10^{-5}\Delta(10)\, \tilde{F}_7(10,1,1) &= -0.16587 \sim -0.16556;\\
10^{-5}\Delta(10)\, \tilde{F}_7(10,1,1.5) &= -0.06049 \sim -0.06016;\\
10^{-5}\Delta(10)\, \tilde{F}_8(10,1,1) &= 0.917943 \sim 0.917881;\\
10^{-5}\Delta(10)\, \tilde{F}_8(10,1,1.5 &= 0.438990 \sim 0.438883;\\
10^{-5}\Delta(10)\, \tilde{F}_9(10,1,1) &= -0.72365 \sim -0.72452;\\
10^{-5}\Delta(10)\, \tilde{F}_9(10,1,1.5) &= -2.71576 \sim -2.71714.
\end{aligned}
\tag{B-66}
$$

14. $\underline{S_{mn}(x)}$

By definition (131),

$$S_{mn}(x) \equiv \frac{1}{m!}\int_0^\infty t^m g^n\left(\frac{t}{x}\right) \sin\left(t - \frac{m\pi}{2}\right) dt;$$

$$m = 0,1,2, \ldots 2n-1; \qquad n = 1,2,3, \ldots . \tag{B-67}$$

The asymptotic expansion for large x of this function is given by

$$S_{mn}(x) \sim \sum_{k=0}^{N_{mn}(x)} s_{mnk} x^{-2k} \tag{B-68}$$

with

$$s_{mnk} \equiv \frac{(n+k-1)!\,(m+2k)!}{(n-1)!\,k!\,m!}. \tag{B-69}$$

The quantity $N_{mn}(x)$ is the largest value of k for which the series in Eq. (B-68) is monotonically decreasing. Since by comparison with Eq. (B-56) we see that

$$s_{mnk} = |q_{mnk}|, \tag{B-70}$$

it follows that the values of $N_{mn}(x)$ are the same in the expansion for Q_{mn} and S_{mn}, for a given m, n, and x. From Eqs. (B-56), (B-57), (B-68), and (B-69), we also observe that

$$S_{mn}(x) \sim Q_{mn}(ix). \tag{B-71}$$

Since in Eqs. (137) $S_{01}(x)$ is given in terms of the exponential integrals $E^*(x)$ and $E_1(x)$, the asymptotic expansions[22] for E^* and E_1 can be substituted into this equation in order to obtain the asymptotic expansion for S_{01}. In this way we can show that Eqs. (B-68) and (B-69) hold for $m = 0$ and $n = 1$. It can then be proven that Eqs. (B-68) and (B-69) are true for arbitrary m and n by induction, making use of the recurrence relations (109) which are valid for S_{mn}.

The expansion given by Eqs. (B-68) and (B-69) can also be obtained by use of the technique discussed in Ref. 14, pp. 63-67.

15. $\underline{W_m(x)}$

From Eq. (160) we have that

$$W_m(x) = Ci(x) - \sum_{j=0}^{m-1} \frac{j!}{x^{j+1}} \sin\left(x - \frac{j\pi}{2}\right), \quad m = 0,1,2, \ldots . \tag{B-72}$$

Reference 22 gives the asymptotic expansion of the cosine integral as

$$Ci(x) \sim \left(\frac{1}{x} - \frac{2!}{x^3} + \frac{4!}{x^5} - \frac{6!}{x^7} + \ldots\right) \sin x$$
$$-\left(\frac{1}{x^2} - \frac{3!}{x^4} + \frac{5!}{x^6} - \frac{7!}{x^8} + \ldots\right) \cos x, \tag{B-73}$$

which can also be written as

$$Ci(x) \sim \sum_{j=0} \frac{j!}{x^{j+1}} \sin\left(x - \frac{j\pi}{2}\right). \tag{B-74}$$

The substitution of Eq. (B-74) into Eq. (B-72) yields

$$W_m(x) \sim \sum_{j=m} \frac{j!}{x^{j+1}} \sin\left(x - \frac{j\pi}{2}\right); \quad m = 0,1,2, \ldots . \tag{B-75}$$

Appendix C. Proof of the Modified Form of Weber's Integral Theorem

In Ref. 13, pp. 468-470, Watson proves the following theorem, attributed to Weber:

Theorem 1:

Weber's Integral Theorem. If $\int_{\epsilon}^{\infty} f(\gamma)\, \gamma d\gamma$ exists and is absolutely convergent, where $\epsilon > 0$, and

$$\Omega_m^0(\gamma,\rho) \equiv Y_m(\gamma)\, J_m(\gamma\rho) - J_m(\gamma)\, Y_m(\gamma\rho), \tag{C-1}$$

then

$$\int_1^{\infty}\int_{\epsilon}^{\infty} \gamma\rho f(\gamma)\, \Omega_m^0(\gamma,\rho)\, \Omega_m^0(\eta,\rho)\, d\gamma d\rho$$

$$= \tfrac{1}{2}[J_m^2(\eta) + Y_m^2(\eta)][f(\eta+0) + f(\eta-0)], \tag{C-2}$$

where η lies inside an interval in which $f(\gamma)$ has limited total fluctuation.

The proof of this thorem in Ref. 13 is based on two lemmas. These are:*

Lemma 1. If $\int_{\epsilon}^{\infty} F(\gamma)\, \gamma^{1/2}\, d\gamma$ exists and is absolutely convergent ($\epsilon > 0$), then

$$\lim_{\lambda\to\infty} \lambda^{1/2} \int_{\epsilon}^{\infty} F(\gamma)\, C_m(\gamma\lambda)\, \gamma d\gamma = 0, \tag{C-3}$$

where the cylinder function C_m is defined by

$$C_m(z) \equiv \sigma[\cos\phi\, J_m(z) + \sin\phi\, Y_m(z)], \tag{C-4}$$

σ and ϕ being constants.

Lemma 2. If $\int_{\epsilon}^{\infty} F(\gamma)\, \gamma^{1/2}\, d\gamma$ exists and is absolutely convergent ($\epsilon > 0$), then

*See Ref. 13, pp. 464-469, for the proofs of these lemmas.

$$\lim_{\lambda\to\infty}\int_{\epsilon}^{\infty} F(\gamma)[\gamma C_{m+1}(\gamma\lambda)\, C_m(\eta\lambda) - \eta C_{m+1}(\eta\lambda)\, C_m(\gamma\lambda)]\, \frac{\lambda\gamma\, d\gamma}{\gamma^2 - \eta^2}$$

$$= \tfrac{1}{2}\,\sigma^2[F(\eta+0) + F(\eta-0)], \qquad \text{(C-5)}$$

provided η lies inside an interval in which $F(\gamma)$ has limited total fluctuation.

In the solution of the problem discussed in this report, an integral representation is employed which is similar in appearance to Eq. (C-2), but which involves the kernel Ω_m, defined by

$$\Omega_m(\gamma,\rho) \equiv Y_m'(\gamma)\, J_m(\gamma\rho) - J_m'(\gamma)\, Y_m(\gamma\rho). \qquad \text{(C-6)}$$

We will prove here the corresponding theorem for this kernel, following closely the proof of Theorem 1 in Ref. 13 and making use of Lemmas 1 and 2.

Theorem 2:

Modified Form of Weber's Integral Theorem. If $\int_{\epsilon}^{\infty} f(\gamma)\,\gamma\, d\gamma$ exists and is absolutely convergent, where $\epsilon > 0$, then

$$\int_1^{\infty}\int_{\epsilon}^{\infty} \gamma\rho f(\gamma)\, \Omega_m(\gamma,\rho)\, \Omega_m(\eta,\rho)\, d\gamma\, d\rho$$

$$= \tfrac{1}{2}\{[J_m'(\eta)]^2 + [Y_m'(\eta)]^2\}\,[f(\eta+0) + f(\eta-0)], \qquad \text{(C-7)}$$

where η lies inside an interval in which $f(\gamma)$ has limited total fluctuation.

The hypotheses of this theorem and Theorem 1 are evidently somewhat more restrictive than is necessary. In particular, in the traditional statement of Weber's Integral Theorem,* the integration limit ϵ is taken to be zero, which requires a restriction on the behavior of $f(\gamma)$ as $\gamma \to 0$. Theorems 1 and 2 are also true for functions $f(\gamma)$ for which $\int_{\epsilon}^{\infty} f(\gamma)\,\gamma\, d\gamma$ does not exist, but the proofs for less restrictive conditions are more lengthy and tedious.** In order to make extensive use of Watson's development, particularly Lemmas 1 and 2, we have retained his hypotheses. In

*See Ref. 26 and Ref. 15, p. 74, for instance.

**See Ref. 11, Appendix B, where Blenkarn proves Theorem 2 for the particular case $m = 0$. In his proof $\epsilon = 0$ and the restrictions on $f(\gamma)$ are that $f(\gamma)/\gamma$ is bounded as $\gamma \to 0$ and $f(\gamma)$ is monotonically decreasing for large γ.

applying Theorem 2 to the problem discussed in this report, we assume that Eq. (C-7) holds (with $\epsilon = 0$) for our unknown function $\gamma^3 D(\gamma)$ [see Eqs. (62), (63)]. The solution of the problem is then independently verified through the governing field equations and boundary conditions, so there is no need to rigorously prove Theorem 2 for the particular case of our solution.

Proof of Theorem 2. Define the auxiliary functions Ω'_m and M:

$$\Omega'_m(\gamma,\rho) \equiv Y'_m(\gamma)\, J'_m(\gamma\rho) - J'_m(\gamma)\, Y'_m(\gamma\rho), \tag{C-8}$$

$$M(\gamma,\eta,\rho) \equiv \eta\rho\Omega_m(\gamma,\rho)\, \Omega'_m(\eta,\rho) - \gamma\rho\Omega_m(\eta,\rho)\, \Omega'_m(\gamma,\rho). \tag{C-9}$$

By Eqs. (C-6) and (C-8),

$$M(\gamma,\eta,1) = 0, \tag{C-10}$$

and, using the recurrence relations for the Bessel functions,[13] we have

$$\frac{\partial M(\gamma,\eta,\rho)}{\partial\rho} = (\gamma^2 - \eta^2)\, \rho\Omega_m(\gamma,\rho)\, \Omega_m(\eta,\rho). \tag{C-11}$$

We conclude from Eqs. (C-10) and (C-11) that

$$M(\gamma,\eta,\lambda) = (\gamma^2 - \eta^2) \int_1^\lambda \rho\Omega_m(\gamma,\rho)\, \Omega_m(\eta,\rho)\, d\rho. \tag{C-12}$$

Define

$$I(\eta) \equiv \lim_{\lambda\to\infty} \int_\epsilon^\infty \gamma f(\gamma) \int_1^\lambda \rho\Omega_m(\gamma,\rho)\, \Omega_m(\eta,\rho)\, d\rho d\gamma. \tag{C-13}$$

The substitution of Eq. (C-12) into Eq. (C-13) yields

$$I(\eta) = \lim_{\lambda\to\infty} \int_\epsilon^\infty \frac{\gamma f(\gamma)\, M(\gamma,\eta,\lambda)}{\gamma^2 - \eta^2}\, d\gamma. \tag{C-14}$$

Replacing Ω_m and Ω'_m in Eq. (C-9) by their definitions (C-6) and (C-8), and using the recurrence relations for the Bessel functions, we have from Eq. (C-14) that

$$I(\eta) = \sum_{j=1}^{5} I_j(\eta), \tag{C-15}$$

where

$$I_1(\eta) \equiv \lim_{\lambda\to\infty} \int_{\epsilon}^{\infty} f(\gamma)\, Y'_m(\gamma)\, Y'_m(\eta)[\gamma J_{m+1}(\gamma\lambda)\, J_m(\eta\lambda) - \eta J_{m+1}(\eta\lambda)\, J_m(\gamma\lambda)] \frac{\lambda\gamma}{\gamma^2 - \eta^2}\, d\gamma;$$

$$I_2(\eta) \equiv \lim_{\lambda\to\infty} \int_{\epsilon}^{\infty} f(\gamma)\, J'_m(\gamma)\, J'_m(\eta)[\gamma Y_{m+1}(\gamma\lambda)\, Y_m(\eta\lambda) - \eta Y_{m+1}(\eta\lambda)\, Y_m(\gamma\lambda)] \frac{\lambda\gamma}{\gamma^2 - \eta^2}\, d\gamma;$$

$$I_3(\eta) \equiv -\lim_{\lambda\to\infty} \int_{\epsilon}^{\infty} \tfrac{1}{4} f(\gamma)[J'_m(\gamma)\, Y'_m(\eta) + J'_m(\eta)\, Y'_m(\gamma)][\gamma P_{m+1}(\gamma\lambda)\, P_m(\eta\lambda)$$

$$- \eta P_{m+1}(\eta\lambda)\, P_m(\gamma\lambda)] \frac{\lambda\gamma}{\gamma^2 - \eta^2}\, d\gamma;$$

$$I_4(\eta) \equiv \lim_{\lambda\to\infty} \int_{\epsilon}^{\infty} \tfrac{1}{4} f(\gamma)[J'_m(\gamma)\, Y'_m(\eta) + J'_m(\eta)\, Y'_m(\gamma)][\gamma Q_{m+1}(\gamma\lambda)\, Q_m(\eta\lambda)$$

$$- \eta Q_{m+1}(\eta\lambda)\, Q_m(\gamma\lambda)] \frac{\lambda\gamma}{\gamma^2 - \eta^2}\, d\gamma;$$

$$I_5(\eta) \equiv \lim_{\lambda\to\infty} \int_{\epsilon}^{\infty} \tfrac{1}{2} f(\gamma)[J'_m(\gamma)\, Y'_m(\eta) - J'_m(\eta)\, Y'_m(\gamma)][\gamma J_{m+1}(\gamma\lambda)\, Y_m(\eta\lambda)$$

$$- \gamma Y_{m+1}(\gamma\lambda)\, J_m(\eta\lambda) + \eta J_{m+1}(\eta\lambda)\, Y_m(\gamma\lambda) - \eta Y_{m+1}(\eta\lambda)\, J_m(\gamma\lambda)] \frac{\lambda\gamma}{\gamma^2 - \eta^2}\, d\gamma, \qquad \text{(C-16)}$$

and the cylinder functions P_m and Q_m are defined by

$$P_m(Z) \equiv J_m(Z) + Y_m(Z);$$

$$Q_m(Z) \equiv J_m(Z) - Y_m(Z). \qquad \text{(C-17)}$$

Since, by hypothesis, $\int_{\epsilon}^{\infty} f(\gamma)\, \gamma\, d\gamma$ exists and is absolutely convergent, the first four integrals in Eqs. (C-16) can each be evaluated by the application of Lemma 2. Table C-I indicates the correspondence between terms in Lemma 2 and the desired integrals.

TABLE C-1. Correspondence between Terms in Lemma 2 and the Integrals $I_j(\eta)$, j = 1,2,3,4

Integral	Terms in (C-4), (C-5)			
	$F(\gamma)$	C_m	σ	ϕ
I_1	$f(\gamma)\, Y'_m(\gamma)\, Y'_m(\eta)$	J_m	1	0
I_2	$f(\gamma)\, J'_m(\gamma)\, J'_m(\eta)$	Y_m	1	$\pi/2$
I_3	$-\frac{1}{4} f(\gamma)[J'_m(\gamma)\, Y'_m(\eta) + J'_m(\eta)\, Y'_m(\gamma)]$	P_m	$\sqrt{2}$	$\pi/4$
I_4	$\frac{1}{4} f(\gamma)[J'_m(\gamma)\, Y'_m(\eta) + J'_m(\eta)\, Y'_m(\gamma)]$	Q_m	$\sqrt{2}$	$-\pi/4$

Therefore, we have

$$I_1(\eta) = \tfrac{1}{2}[Y'_m(\eta)]^2\,[f(\eta+0) + f(\eta-0)];$$

$$I_2(\eta) = \tfrac{1}{2}[J'_m(\eta)]^2\,[f(\eta+0) + f(\eta-0)];$$

$$I_3(\eta) = -\tfrac{1}{2}J'_m(\eta)\,Y'_m(\eta)[f(\eta+0) + f(\eta-0)];$$

$$I_4(\eta) = \tfrac{1}{2}J'_m(\eta)\,Y'_m(\eta)[f(\eta+0) + f(\eta+0)], \tag{C-18}$$

provided η lies inside an interval in which $f(\gamma)$ has limited total fluctuation.

The integral $I_5(\eta)$ can be expressed as the sum of four integrals each of which can be evaluated using Lemma 1. Only the first of these will be discussed here, since they can all be treated in the same manner. Consider

$$I_{51}(\eta) \equiv \lim_{\lambda\to\infty}\int_{\epsilon}^{\infty} \tfrac{1}{2}f(\gamma)[J'_m(\gamma)\,Y'_m(\eta) - J'_m(\eta)\,Y'_m(\gamma)]$$

$$[\gamma J_{m+1}(\gamma\lambda)\,Y_m(\eta\lambda)]\,\frac{\lambda\gamma}{\gamma^2-\eta^2}\,d\gamma, \tag{C-19}$$

which is equivalent to

$$I_{51}(\eta) = \lim_{\lambda\to\infty}[\sqrt{\lambda}\,Y_m(\eta\lambda)]\,\lim_{\lambda\to\infty}\sqrt{\lambda}\int_{\epsilon}^{\infty}\overline{F}(\gamma)\,J_{m+1}(\gamma\lambda)\,\gamma\,d\gamma, \tag{C-20}$$

where

$$\overline{F}(\gamma) \equiv \tfrac{1}{2}f(\gamma)\,\frac{\gamma}{\gamma^2-\eta^2}\,[J'_m(\gamma)\,Y'_m(\eta) - J'_m(\eta)\,Y'_m(\gamma)]. \tag{C-21}$$

The first limit in Eq. (C-20) is bounded because of the behavior of the Bessel function for large values of its argument. The second limit will be zero by Lemma 1 if $\overline{F}(\gamma)$ meets the hypotheses of this lemma. Since $[J'_m(\gamma)\,Y'_m(\eta) - J'_m(\eta)\,Y'_m(\gamma)]/(\gamma^2-\eta^2)$ is bounded for γ near η, as can be shown by L'Hospital's Rule, $\int_{\epsilon}^{\infty}\overline{F}(\gamma)\sqrt{\gamma}\,d\gamma$ exists and is absolutely convergent if $\int_{\epsilon}^{\infty} f(\gamma)\,\gamma\,d\gamma$ exists and is absolutely convergent.

Consequently,

$$I_{51}(\eta) = I_5(\eta) = 0. \tag{C-22}$$

By Eqs. (C-15) and (C-18), and the above discussion of I_5 we have, finally, that

$$I(\eta) = \frac{1}{2}\{[J'_m(\eta)]^2 + [Y'_m(\eta)]^2\}\,[f(\eta+0) + f(\eta-0)], \tag{C-23}$$

or, by the definition (C-13) of $I(\eta)$,

$$\int_1^\infty \int_\epsilon^\infty \gamma\rho f(\gamma)\,\Omega_m(\gamma,\rho)\,\Omega_m(\eta,\rho)\,d\gamma\,d\rho$$
$$= \frac{1}{2}\{[J'_m(\eta)]^2 + [Y'_m(\eta)]^2\}\,[f(\eta+0) + f(\eta-0)] \tag{C-24}$$

which is what was to be shown.

Appendix D. Computer Programs

The computer programs, which were written in FORTRAN for the IBM-704, comprise three groups: Programs 8 and 8A solve the integral equation (65) for the function $D(\eta)$ at 75 points in the interval (0,2000) for $\alpha = 1$ and $3/2$, respectively. The boundary condition on τ_{zz}^{R} at $z = 0$ was checked and the displacement u_z^R on $z = 0$ was calculated through the use of Programs 15, 16, 9, 10A, 14, and 11. Finally, the stresses τ_{zz}^{R}, $\tau_{\theta\theta}^{R}$, and $\tau_{\theta z}^{R}$ on the hole $r = a$ were calculated by the successive applications of Programs 7, 3A, and 17.

In addition, Programs 1, 2, 3, 4, 5, 6, and 10, which are not reproduced here, were devised to calculate some of the needed auxiliary functions, such as T_{zz}^{0}, $\tilde{T}_{zz}$, Ω_2, Q_{mn}, and Δ. After being checked to see that the functions were computed correctly and accurately, these programs were then incorporated into the programs discussed above.

```
C      PROGRAM 8. SOLUTION OF INTEGRAL EQUATION FOR
C      THE FUNCTIONG(XI). ALPHA IS 1.0 (POISSON RATIO 0.5).
C      G IS DETERMINED AT 75 POINTS (.25,5,.25),(5,10,.5),
C      (10,20,1),(20,100,5),(100,2000,100).
C      L IS KERNEL OF INTEGRAL EQUATION
C      LO IS EXACT FUNCTION OF XI, ETA
C      LS IS INTEGRAL OVER GAMMA(0,10)BY SIMPSONS RULE
C      LL IS INTEGRAL OVER GAMMA (10,INF) BY ASYMPTOTIC METHODS
       DIMENSIONETA(75),A(75,75),TABR(66),Q1(75),Q2(75),
      XQ3(75),Q4(75),TABJ(66),TABY(66),B(75,1)
       PI=3.14159265
       CLIM=1.0E-8
       CST1=1.03204910
C      STORING OF ETAS
       DO801IE=1,20
  801  ETA(IE)=0.25*FLOATF(IE)
       DO802IE=21,30
  802  ETA(IE)=-5.+0.5*FLOATF(IE)
       DO803IE=31,40
  803  ETA(IE)=IE-20
       DO804IE=41,56
  804  ETA(IE)=5*IE-180
       DO805IE=57,75
  805  ETA(IE)=100*IE-5500
C      CALCULATION OF LS(XI,ETA)
C      INITIALIZATION PLUS VALUE FOR GAMMA EQUAL 10
       JG=1
       IG=100
       GOTO806
C      INTEGRATION FOR GAMMA(.1,9.9,.1)
C      GAMMA ROUTING
  813  JG=2
       IG=1
       GOTO806
  815  IG=IG+2
       IF(98-IG)816,818,806
  816  JG=3
       IG=2
       GOTO806
  818  JG=4
       GOTO806
C      SUBROUTINE FOR F1, F2, F3
  806  G=0.1*FLOATF(IG)
       Z1=BESKF(G,0.,1,66,XLOCF(TABR))
       AK0=TABR(1)
       AK1=TABR(2)
       G2=G*G
       AK=-(2.+(G2*AK1)/(G*AK0+2.*AK1))
       AKSQ=AK*AK
       FFG=G2*(G2+8.-2.*AK)+12.
       FFG1=G2+4.-AKSQ
       DELG2=(FFG1*FFG-4.*G2+AK*(8.*G2+24.
      X-G2*AK-6.*AKSQ))/G2
       F1=1.-(G2*FFG1+3.*AKSQ)/DELG2
       F2=2.*(G-1.+AK*(-G2+(G2+3.)*AK)/DELG2)
       F3=-G*(G2+4.875)+2.5-AKSQ*FFG/DELG2
C      FUNCTIONS OF GAMMA, ETA, XI
```

```
      IX=1
  810 X=ETA(IX)
      SGX=X*X/(X*X+G2)
      IE=1
  811 E=ETA(IE)
      SGE=E*E/(E*E+G2)
      DLSIN=(SGX*SGE)*(F1+(SGX+SGE)*F2+(SGX*SGE)*F3)
      GOTO(807,814,817,819),JG
C     XI, ETA ROUTING
  808 IE=IE+1
      IF(IX-IE)809,811,811
  809 IX=IX+1
      IF(75-IX)812,810,810
  812 GOTO(813,815,815,820),JG
C     SUMMATIONS
  807 A(IX,IE)=0.5+DLSIN
      GOTO808
  814 A(IX,IE)=A(IX,IE)+4.*DLSIN
      GOTO808
  817 A(IX,IE)=A(IX,IE)+2.*DLSIN
      GOTO808
  819 A(IX,IE)=(A(IX,IE)+2.*DLSIN)/30.
      GOTO808
  820 WRITEOUTPUTTAPE6,887
C     FUNCTIONS OF ETA
C     ASYMPTOTIC EXPANSION COEFFICIENTS FOR F1, F2, F3
      B11=6.
      B12=-0.3
      B13=-0.3675
      B14=0.09225
      B15=181.828125E-4
      B16=-1403.4375E-5
      B17=1676.0566E-6
      B18=9726.75E-7
      B21=-0.75
      B22=-1.05
      B23=-.12984375
      B24=.210375
      B25=-498.14648E-4
      B26=-782.4375E-5
      B31=11.5546875
      B32=-4.63125
      B33=1.2664746
      B34=-8.5781251E-3
C     5F1(10),5F2(10),5F3(10)
      F1A=2.7155861
      F2A=-.88562005
      F3A=4.0464440
C     CALCULATION OF H FUNCTIONS
      DO825IE=1,75
      E=ETA(IE)
      E2=E*E
      EA=E/10.
      EA2=E2/100.
      EEAA=E2/(100.+E2)
      EEAA2=EEAA*EEAA
      IF(8.-E)860,861,861
C     SUBROUTINE FOR H FUNCTIONS FOR SMALL ETA
  861 ZN7=4.
      ZN8=9.
      EA2N=EA2
      H7=0.0
```

```
      H8=0.0
  863 H7=H7+EA2N/ZN7
      H8=H8+EA2N/ZN8
      ABEN=ABSF(EA2N)
      IF(CLIM-ABEN)862,862,864
  862 EA2N=-EA2N*EA2
      ZN7=ZN7+1.
      ZN8=ZN8+2.
      GOTO863
  864 H5=EA2*(.333333333-H7)
      H3=EA2*(0.5-H5)
      H1=EA2*(1.-H3)
      H6=EA2*(.142857143-H8)
      H4=EA2*(0.2-H6)
      H2=EA2*(.333333333-H4)
      GOTO865
C     H NOT SMALL
  860 H1=LOGF(1.+EA2)
      AT=ATANF(EA)
      H2=1.-AT/EA
      H3=1.-H1/EA2
      H4=.333333333-H2/EA2
      H5=0.5-H3/EA2
      H6=0.2-H4/EA2
      H7=.333333333-H5/EA2
      H8=.142857143-H6/EA2
  865 CONTINUE
C     CALCULATION OF Q FUNCTIONS
      Q1(IE)=((0.5*B11+B21)*H1+(B12+2.5*B22)*H2+(0.5*B13
     X+1.5*B23)*H3+(B14+3.5*B24)*H4+(0.5*B15+2.*B25)*H5
     X+(B16+4.5*B26)*H6+0.5*B17*H7+B18*H8-F2A*
     XEEAA)/E2
      Q2(IE)=(B31*H1+2.*B32*H2+B33*H3+2.*B34*H4)/E2
      Q3(IE)=(0.5*B31*H1+1.5*B32*H2+B33*H3+2.5*B34*H4
     X-F3A*EEAA)/(E2*E2)
      Q4(IE)=-F1A*EEAA+0.5*B11*H1+1.5*B12*H2+B13*H3
     X+2.5*B14*H4+1.5*B15*H5+3.5*B16*H6+2.*B17*H7
     X+4.5*B18*H8-F2A*EEAA2+B21*(H1-EEAA)
     X+3.75*B22*(H2-EEAA/3.)+3.*B23*(H3-0.5*EEAA)
     X+8.75*B24*(H4-0.2*EEAA)+6.*B25*(H5-EEAA/3.)
     X+15.75*B26*(H6-EEAA/7.)-F3A*EEAA*EEAA2/3.
     X+0.5*B31*(H1-EEAA-0.5*EEAA2)+2.1875*B32*(H2
     X-EEAA/3.-EEAA2/7.5)+2.*B33*(H3-0.5*EEAA-EEAA2/6.)
     X+6.5625*B34*(H4-0.2*EEAA-EEAA2/17.5)
      WRITEOUTPUTTAPE6,888,E,H1,H2,H3,H4,H5,H6,H7,H8,
     1Q1(IE),Q2(IE),Q3(IE),Q4(IE)
  825 CONTINUE
C     CALCULATION OF LO(XI,ETA),LL(XI,ETA)
C     DIAGONAL ELEMENTS
      WRITEOUTPUTTAPE6,880
      DO830IE=1,75
      E=ETA(IE)
      DLO=(.109375*PI+E*(-.1875+E*E/12.))*E
      DLL=Q4(IE)
      DLS=A(IE,IE)
      DL=CST1*(DLS+DLL+DLO)
      WRITE OUTPUT TAPE6,883,E,DLO,DLS,DLL,DL
  830 A(IE,IE)=DL/3.
C     OFF-DIAGONAL ELEMENTS
      WRITEOUTPUTTAPE6,890
      DO831IX=2,75
      X=ETA(IX)
```

```
      X2=X*X
      WRITEOUTPUTTAPE6,884,X
      NE=IX-1
      DO831IE=1,NE
      E=ETA(IE)
      E2=E*E
      DXE=(X-E)*(X+E)
      BLGN=LOGF(X/E)
      SXED=X2*E2/DXE
      XED=X*E/(X+E)
      DL0=0.125*PI*XED*(3.-5.*XED/(X+E))-2.*SXED*BLGN
     X+SXED*(SXED*BLGN*(X2+E2-9.75)-X2*E2
     X+2.4375*(X2+E2))/DXE
      DLL=SXED*(Q1(IE)-Q1(IX)+SXED*((Q2(IX)-Q2(IE))
     X/DXE+Q3(IX)+Q3(IE)))
      DLS=A(IX,IE)
      DL=CST1*(DL0+DLL+DLS)
      WRITEOUTPUTTAPE6,883,E,DL0,DLS,DLL,DL
C     COMPLETION OF L(XI,ETA)MATRIX AND WEIGHTING FACTORS
      A(IX,IE)=DL/3.
  831 A(IE,IX)=DL/3.
      DO835IX=1,75
      DO837IE=2,18,2
  837 A(IX,IE)=0.5*A(IX,IE)
      A(IX,20)=0.75*A(IX,20)
      DO838IE=21,29,2
  838 A(IX,IE)=2.*A(IX,IE)
      A(IX,30)=1.5*A(IX,30)
      DO839IE=31,39,2
  839 A(IX,IE)=4.*A(IX,IE)
      A(IX,32)=2.*A(IX,32)
      A(IX,34)=2.*A(IX,34)
      A(IX,36)=2.*A(IX,36)
      A(IX,38)=2.*A(IX,38)
      A(IX,40)=6.*A(IX,40)
      DO841IE=41,55,2
  841 A(IX,IE)=20.*A(IX,IE)
      DO842IE=42,54,2
  842 A(IX,IE)=10.*A(IX,IE)
      A(IX,56)=105.*A(IX,56)
      DO843IE=57,75,2
  843 A(IX,IE)=400.*A(IX,IE)
      DO844IE=58,74,2
  844 A(IX,IE)=200.*A(IX,IE)
  835 CONTINUE
C     CALCULATION OF Y(XI)
      WRITEOUTPUTTAPE6,891
C     XI LESS THAN 10
      DO850IX=1,30
      X=ETA(IX)
      X2=X*X
      Z2=BESJF(X,0.,1,66,XLOCF(TABJ))
      Z3=BESYF(X,0.,1,66,XLOCF(TABY))
      AJ0=TABJ(1)
      AJ1=TABJ(2)
      AY0=TABY(1)
      AY1=TABY(2)
      EJ2PR=(X-4./X)*AJ1+2.*AJ0
      EY2PR=(X-4./X)*AY1+2.*AY0
      Y=X2*X2*(EJ2PR*EJ2PR+EY2PR*EY2PR)
      WRITEOUTPUTTAPE6,885,X,Y
  850 A(IX,IX)=A(IX,IX)-Y
```

```
C     XI GREATER THAN 10
      DO851IX=31,75
      X=ETA(IX)
      X2=X*X
      X4=X2*X2
      Y=.63661977*X*(X4-1.625*X2+3.3984375+16.3037109/X2
     X-61.283112/X4)
      WRITEOUTPUTTAPE6,885,X,Y
  851 A(IX,IX)=A(IX,IX)-Y
      WRITEOUTPUTTAPE6,892
C     LOAD MATRIX B
      DO853IX=1,75
  853 B(IX,1)=-1.27323954
C     SOLUTION OF SYSTEM OF ALGEBRAIC EQUATIONS FOR THE
C     FUNCTION G(XI) USING SUBROUTINE MATINV
      CALLMATINV(A,75,B,1,DETERM)
      WRITEOUTPUTTAPE6,885,(ETA(IX),B(IX,1),IX=1,75)
C     FORMATS
  887 FORMAT(30H1ETA, H FUNCTIONS, Q FUNCTIONS)
  888 FORMAT(F7.3/6E17.8/6E17.8)
  880 FORMAT(44H1SOLUTION OF INTEGRAL EQUATION FOR ALPHA 1.0//18H DIAGON
     XAL ELEMENTS/19H ETA, LO, LS, LL, L)
  883 FORMAT(F10.3,4E20.8)
  890 FORMAT(22H1OFF-DIAGONAL ELEMENTS/19H ETA, LO, LS, LL, L)
  884 FORMAT(7H XI IS F10.3)
  891 FORMAT(10H1XI, Y(XI))
  885 FORMAT(F10.3,E20.8)
  892 FORMAT(31H1XI, G(XI) FOR ALPHA EQUALS 1.0)
      CALL SYSTEM
      END (  1 ,  1 ,  0 ,  1 ,  0  )                          ANL-30
```

```
C     PROGRAM 8A. SOLUTION OF INTEGRAL EQUATION FOR
C     THE FUNCTION G(XI). ALPHA IS 1.5 (POISSON RATIO 0.25).
C     G IS DETERMINED AT 75 POINTS (.25,5,.25),(5,10,.5),
C     (10,20,1),(20,100,5),(100,2000,100).
C     L IS KERNEL OF INTEGRAL EQUATION
C     LO IS EXACT FUNCTION OF XI, ETA
C     LS IS INTEGRAL OVER GAMMA(0,10)BY SIMPSONS RULE
C     LL IS INTEGRAL OVER GAMMA (10,INF) BY ASYMPTOTIC METHODS
      DIMENSIONETA(75),A(75,75),TABR(66),Q1(75),Q2(75),
     XQ3(75),Q4(75),TABJ(66),TABY(66),B(75,1)
      PI=3.14159265
      CLIM=1.0E-8
      CST1=1.54807365
C     STORING OF ETAS
      DO801IE=1,20
  801 ETA(IE)=0.25*FLOATF(IE)
      DO802IE=21,30
  802 ETA(IE)=-5.+0.5*FLOATF(IE)
      DO803IE=31,40
  803 ETA(IE)=IE-20
      DO804IE=41,56
  804 ETA(IE)=5*IE-180
      DO805IE=57,75
  805 ETA(IE)=100*IE-5500
C     CALCULATION OF LS(XI,ETA)
C     INITIALIZATION PLUS VALUE FOR GAMMA EQUAL 10
      JG=1
      IG=100
      GOTO806
C     INTEGRATION FOR GAMMA(.1,9.9,.1)
C     GAMMA ROUTING
  813 JG=2
      IG=1
      GOTO806
  815 IG=IG+2
      IF(98-IG)816,818,806
  816 JG=3
      IG=2
      GOTO806
  818 JG=4
      GOTO806
C     SUBROUTINE FOR F1, F2, F3
  806 G=0.1*FLOATF(IG)
      Z1=BESKF(G,0.,1,66,XLOCF(TABR))
      AK0=TABR(1)
      AK1=TABR(2)
      G2=G*G
      AK=-(2.+(G2*AK1)/(G*AK0+2.*AK1))
      AKSQ=AK*AK
      FFG=G2*(G2+8.-2.*AK)+12.
      FFG1=G2+4.-AKSQ
      DELG2=(FFG1*FFG-6.*G2+AK*(12.*G2+36.
     X-1.5*G2*AK-9.*AKSQ))/G2
      F1=1.-(G2*FFG1+4.5*AKSQ)/DELG2
      F2=2.*G-3.+2.*AK*(-G2+(G2+3.)*AK)/DELG2
      F3=G*(-G2+0.5*G-5.125)/1.5+1.-AKSQ*FFG/(1.5*DELG2)
C     FUNCTIONS OF GAMMA, ETA, XI
```

```
      IX=1
  810 X=ETA(IX)
      SGX=X*X/(X*X+G2)
      IE=1
  811 E=ETA(IE)
      SGE=E*E/(E*E+G2)
      DLSIN=(SGX*SGE)*(F1+(SGX+SGE)*F2+(SGX*SGE)*F3)
      GOTO(807,814,817,819),JG
C     XI, ETA ROUTING
  808 IE=IE+1
      IF(IX-IE)809,811,811
  809 IX=IX+1
      IF(75-IX)812,810,810
  812 GOTO(813,815,815,820),JG
C     SUMMATIONS
  807 A(IX,IE)=DLSIN-3.83333333
      GOTO808
  814 A(IX,IE)=A(IX,IE)+4.*DLSIN
      GOTO808
  817 A(IX,IE)=A(IX,IE)+2.*DLSIN
      GOTO808
  819 A(IX,IE)=(A(IX,IE)+2.*DLSIN)/30.
      GOTO808
  820 WRITEOUTPUTTAPE6,887
C     FUNCTIONS OF ETA
C     ASYMPTOTIC EXPANSION COEFFICIENTS FOR F1, F2, F3
C     ALPHA1.5
      B11=8.
      B12=-1.
      B13=-.355
      B14=.16325
      B15=.0011
      B16=-.0190075
      B17=5257.91797E-6
      B18=5876.75E-7
      B21=-2.25
      B22=-1.65
      B23=.19453125
      B24=.2094375
      B25=-925.353516E-4
      B26=431.0625E-5
      B31=2.515625
      B32=-3.00625
      B33=1.53451172
      B34=-.23175
C     5F1(10),5F2(10),5F3(10)
      F1A=3.39780533
      F2A=-1.78905591
      F3A=.381009667
C     CALCULATION OF H FUNCTIONS
      DO825IE=1,75
      E=ETA(IE)
      E2=E*E
      EA=E/10.
      EA2=E2/100.
      EEAA=E2/(100.+E2)
      EEAA2=EEAA*EEAA
      IF(8.-E)860,861,861
C     SUBROUTINE FOR H FUNCTIONS FOR SMALL ETA
  861 ZN7=4.
      ZN8=9.
      EA2N=EA2
```

```
      H7=0.0
      H8=0.0
  863 H7=H7+EA2N/ZN7
      H8=H8+EA2N/ZN8
      ABEN=ABSF(EA2N)
      IF(CLIM-ABEN)862,862,864
  862 EA2N=-EA2N*EA2
      ZN7=ZN7+1.
      ZN8=ZN8+2.
      GOTO863
  864 H5=EA2*(.333333333-H7)
      H3=EA2*(0.5-H5)
      H1=EA2*(1.-H3)
      H6=EA2*(.142857143-H8)
      H4=EA2*(0.2-H6)
      H2=EA2*(.333333333-H4)
      GOTO865
C     H NOT SMALL
  860 H1=LOGF(1.+EA2)
      AT=ATANF(EA)
      H2=1.-AT/EA
      H3=1.-H1/EA2
      H4=.333333333-H2/EA2
      H5=0.5-H3/EA2
      H6=0.2-H4/EA2
      H7=.333333333-H5/EA2
      H8=.142857143-H6/EA2
  865 CONTINUE
C     CALCULATION OF Q FUNCTIONS
      Q1(IE)=((0.5*B11+B21)*H1+(B12+2.5*B22)*H2+(0.5*B13
     X+1.5*B23)*H3+(B14+3.5*B24)*H4+(0.5*B15+2.*B25)*H5
     X+(B16+4.5*B26)*H6+0.5*B17*H7+B18*H8-F2A*
     XEEAA)/E2
      Q2(IE)=(B31*H1+2.*B32*H2+B33*H3+2.*B34*H4)/E2
      Q3(IE)=(0.5*B31*H1+1.5*B32*H2+B33*H3+2.5*B34*H4
     X-F3A*EEAA)/(E2*E2)
      Q4(IE)=-F1A*EEAA+0.5*B11*H1+1.5*B12*H2+B13*H3
     X+2.5*B14*H4+1.5*B15*H5+3.5*B16*H6+2.*B17*H7
     X+4.5*B18*H8-F2A*EEAA2+B21*(H1-EEAA)
     X+3.75*B22*(H2-EEAA/3.)+3.*B23*(H3-0.5*EEAA)
     X+8.75*B24*(H4-0.2*EEAA)+6.*B25*(H5-EEAA/3.)
     X+15.75*B26*(H6-EEAA/7.)-F3A*EEAA*EEAA2/3.
     X+0.5*B31*(H1-EEAA-0.5*EEAA2)+2.1875*B32*(H2
     X-EEAA/3.-EEAA2/7.5)+2.*B33*(H3-0.5*EEAA-EEAA2/6.)
     X+6.5625*B34*(H4-0.2*EEAA-EEAA2/17.5)
      WRITEOUTPUTTAPE6,888,E,H1,H2,H3,H4,H5,H6,H7,H8,
     1Q1(IE),Q2(IE),Q3(IE),Q4(IE)
  825 CONTINUE
C     CALCULATION OF LO(XI,ETA),LL(XI,ETA)
C     DIAGONAL ELEMENTS
      WRITEOUTPUTTAPE6,880
      DO830IE=1,75
      E=ETA(IE)
      DLO=E*PI*(.71875-(E*E)/96.)+(E*E)*((E*E)/18.-.430555556)
      DLL=Q4(IE)
      DLS=A(IE,IE)
      DL=CST1*(DLS+DLL+DLO)
      WRITE OUTPUT TAPE6,883,E,DLO,DLS,DLL,DL
  830 A(IE,IE)=DL/3.
C     OFF-DIAGONAL ELEMENTS
      WRITEOUTPUTTAPE6,890
      DO831IX=2,75
```

```
      X=ETA(IX)
      X2=X*X
      WRITEOUTPUTTAPE6,884,X
      NE=IX-1
      DO831IE=1,NE
      E=ETA(IE)
      E2=E*E
      DXE=(X-E)*(X+E)
      BLGN=LOGF(X/E)
      SXED=X2*E2/DXE
      XED=X*E/(X+E)
      DL0=PI*XED*(1.5-XED*(.25/(X+E)+XED/12.))
     X-2.*SXED*BLGN+(SXED*SXED/1.5)*((X2+E2)*BLGN/DXE
     X-1.)+(41.*SXED/(24.*DXE))*(X2+E2-4.*SXED*BLGN)
      DLL=SXED*(Q1(IE)-Q1(IX)+SXED*((Q2(IX)-Q2(IE))
     X/DXE+Q3(IX)+Q3(IE)))
      DLS=A(IX,IE)
      DL=CST1*(DL0+DLL+DLS)
      WRITEOUTPUTTAPE6,883,E,DL0,DLS,DLL,DL
C     COMPLETION OF L(XI,ETA)MATRIX AND WEIGHTING FACTORS
      A(IX,IE)=DL/3.
  831 A(IE,IX)=DL/3.
      DO835IX=1,75
      DO837IE=2,18,2
  837 A(IX,IE)=0.5*A(IX,IE)
      A(IX,20)=0.75*A(IX,20)
      DO838IE=21,29,2
  838 A(IX,IE)=2.*A(IX,IE)
      A(IX,30)=1.5*A(IX,30)
      DO839IE=31,39,2
  839 A(IX,IE)=4.*A(IX,IE)
      A(IX,32)=2.*A(IX,32)
      A(IX,34)=2.*A(IX,34)
      A(IX,36)=2.*A(IX,36)
      A(IX,38)=2.*A(IX,38)
      A(IX,40)=6.*A(IX,40)
      DO841IE=41,55,2
  841 A(IX,IE)=20.*A(IX,IE)
      DO842IE=42,54,2
  842 A(IX,IE)=10.*A(IX,IE)
      A(IX,56)=105.*A(IX,56)
      DO843IE=57,75,2
  843 A(IX,IE)=400.*A(IX,IE)
      DO844IE=58,74,2
  844 A(IX,IE)=200.*A(IX,IE)
  835 CONTINUE
C     CALCULATION OF Y(XI)
      WRITEOUTPUTTAPE6,891
C     XI LESS THAN 10
      DO850IX=1,30
      X=ETA(IX)
      X2=X*X
      Z2=BESJF(X,0.,1,66,XLOCF(TABJ))
      Z3=BESYF(X,0.,1,66,XLOCF(TABY))
      AJ0=TABJ(1)
      AJ1=TABJ(2)
      AY0=TABY(1)
      AY1=TABY(2)
      EJ2PR=(X-4./X)*AJ1+2.*AJ0
      EY2PR=(X-4./X)*AY1+2.*AY0
      Y=X2*X2*(EJ2PR*EJ2PR+EY2PR*EY2PR)
      WRITEOUTPUTTAPE6,885,X,Y
```

```
850 A(IX,IX)=A(IX,IX)-Y
C     XI GREATER THAN 10
      DO851IX=31,75
      X=ETA(IX)
      X2=X*X
      X4=X2*X2
      Y=.63661977*X*(X4-1.625*X2+3.3984375+16.3037109/X2
     X-61.283112/X4)
      WRITEOUTPUTTAPE6,885,X,Y
  851 A(IX,IX)=A(IX,IX)-Y
      WRITEOUTPUTTAPE6,892
C     LOAD MATRIX B
      DO853IX=1,75
  853 B(IX,1)=-.636619772
C     SOLUTION OF SYSTEM OF ALGEBRAIC EQUATIONS FOR THE
C     FUNCTION G(XI) USING SUBROUTINE MATINV
      CALLMATINV(A,75,B,1,DETERM)
      WRITEOUTPUTTAPE6,885,(ETA(IX),B(IX,1),IX=1,75)
C     FORMATS
  887 FORMAT(30H1ETA, H FUNCTIONS, Q FUNCTIONS)
  888 FORMAT(F7.3/6E17.8/6E17.8)
  880 FORMAT(44H1SOLUTION OF INTEGRAL EQUATION FOR ALPHA 1.5//18H DIAGON
     1AL ELEMENTS/19H ETA, LO, LS, LL, L)
  883 FORMAT(F10.3,4E20.8)
  890 FORMAT(22H1OFF-DIAGONAL ELEMENTS/19H ETA, LO, LS, LL, L)
  884 FORMAT(7H XI IS F10.3)
  891 FORMAT(10H1XI, Y(XI))
  885 FORMAT(F10.3,E20.8)
  892 FORMAT(31H1XI, G(XI) FOR ALPHA EQUALS 1.5)
      CALL SYSTEM
      END  (    1 ,    1 ,    0 ,    1 ,    0    )                ANL-30
```

```
C       PROGRAM 15. BOUNDARY CONDITION.
C       INTEGRALS OF G(ALPHA, ETA) X BC(ETA, RHO)
C       BC ARE INTEGRALS FROM 0 TO 10
C       BASED ON PROGRAM 5.
        DIMENSIONRHO(15),ETA(40),BC1(40,15),BC15(40,15),CR(9,15)
        DIMENSIONG(2,40)
        DIMENSIONTABR(66),TABG(66)
C       INPUT OF RHOS
        READINPUTTAPE7,580,NR,(RHO(IR),IR=1,NR)
  580   FORMAT(I2/(F12.3))
C       STORING OF ETAS
        DO1501IE=1,20
 1501   ETA(IE)=0.25*FLOATF(IE)
        DO1502IE=21,30
 1502   ETA(IE)=-5.0+0.5*FLOATF(IE)
        DO1503IE=31,40
 1503   ETA(IE)=IE-20
        CST=2.7018982E-2
C       FUNCTIONS OF RHO
        DO503IR=1,NR
        R=RHO(IR)
        CR(9,IR)=1./SQRTF(R)
        R1=(R-1.)/R
        R2=R1*(R+1.)/R
        R3=R1*(R*R+R+1.)/(R*R)
        AL1=-1.875*R1
        AL2=-.8203125*R2+3.515625*R1
        AL3=.30761719*R3+1.5380859*R2-5.0537109*R1
        ALS1=-2.375*R1
        ALS2=-3.6328125*R2+5.640625*R1
        ALS3=-1.7431641*R3+8.6279279*R2-4.7685547*R1
        CR(1,IR)=AL1+R*(1.-ALS1)
        CR(2,IR)=-0.5+AL1+R*(1.5-ALS1)
        CR(3,IR)=-AL1-1.+R*ALS1
        CR(4,IR)=-AL1-0.5+R*(ALS1-0.5)
        CR(5,IR)=-AL2-AL1+4.875*(R-1.)+R*ALS2
        CR(6,IR)=-AL2-0.5*AL1-4.625+R*(ALS2-0.5*ALS1+5.125)
        CR(7,IR)=-AL3-AL2-4.875*AL1+R*(ALS3+4.875*ALS1-2.5)
        CR(8,IR)=-AL3-0.5*AL2-4.625*AL1+0.75+R*(ALS3
       X-0.5*ALS2+5.125*ALS1-1.5)
  503   CONTINUE
C       INITIALIZATION OF BC1, BC15
        DO504IR=1,NR
        DO505IE=1,40
        R=RHO(IR)
        BC1(IE,IR)=-1./(R*R)-CR(9,IR)*(2.*CR(1,IR)+CR(7,IR))
  505   BC15(IE,IR)=-0.25/(R*R)-CR(9,IR)*(3.*CR(2,IR)+CR(8,IR))
  504   CONTINUE
C       GAMMA ROUTING
        JG=1
        IG=1
        GOTO513
  506   IG=IG+2
        IF(100-IG)507,508,513
  507   JG=2
        IG=2
```

```
      GOTO513
  508 JG=3
      GOTO513
C     ETA LOOPING
  509 IE=IE+1
      IF(40-IE)510,511,511
C     RHO LOOPING
  510 IR=IR+1
      IF(NR-IR)512,514,514
  512 GOTO(506,506,515),JG
C     FUNCTIONS OF GAMMA IN INTEGRANDS
  513 G=0.1*FLOATF(IG)
      GSQ=G*G
      Z1=BESKF(G,0.,1,66,XLOCF(TABR))
      AK0=TABR(1)
      AK1=TABR(2)
      AK=-(2.+(GSQ*AK1)/(G*AK0+2.*AK1))
      AKSQ=AK*AK
      EK2=AK0+2.*AK1/G
      DELTA1=(GSQ+4.-AKSQ)*(GSQ*(GSQ+8.-2.*AK)+12.)
      DELTA2=-4.*GSQ+AK*(8.*GSQ+24.-GSQ*AK-6.*AKSQ)
      DEK1=EK2*(DELTA1+DELTA2)
      DEK15=EK2*(DELTA1+1.5*DELTA2)
      FG=GSQ*(GSQ*GSQ+8.*GSQ+12.-2.*GSQ*AK)
      HG=GSQ*(GSQ-(GSQ+3.)*AK)
      A41=GSQ*(-GSQ*GSQ-9.*GSQ-12.+(3.*GSQ+3.)*AK)/DEK1
      A415=GSQ*(-GSQ*GSQ-9.*GSQ-12.+(3.*GSQ+1.5)*AK)/DEK15
      AB41=-HG/DEK1
      AB415=-HG/DEK15
      A51=((GSQ+4.-2.*AK)*FG-4.*HG)/DEK1
      A515=((GSQ+4.-2.*AK)*FG-6.*HG)/DEK15
      AB51=-AK*FG/DEK1
      AB515=-AK*FG/DEK15
C     FUNCTIONS OF GAMMA, RHO IN INTEGRAND
      IR=1
  514 R=RHO(IR)
      GR=G*R
      IF(10.-GR)516,517,517
  517 Z2=BESKF(GR,0.,1,66,XLOCF(TABG))
      AKR0=TABG(1)
      AKR1=TABG(2)
      EKR2=AKR0+2.*AKR1/GR
      EKRP=-(2.*AKR0+(GR+4./GR)*AKR1)
      GOTO518
C     ASYMPTOTIC EXPANSIONS FOR EKR2, EKRP
  516 RT1=SQRTF(1.5707963/GR)
      RT2=-SQRTF(1.5707963*GR)
      GR2=GR*GR
      GR3=GR*GR2
      GR4=GR2*GR2
      EKR2=RT1*(1.+1.875/GR+.8203125/GR2-.30761719/GR3
     X+.31723022/GR4-.51549912/(GR2*GR3)
     X+1.1276543/(GR3*GR3)-3.0809127/(GR4*GR3))
      EKRP=RT2*(1.+2.375/GR+3.6328125/GR2+1.7431641/GR3
     X-.75942993/GR4+.91203690/(GR2*GR3)
     X-1.7075908/(GR3*GR3)+4.2488404/(GR3*GR4))
      GOTO518
C     CALCULATION OF H4, H5
  518 H41=CR(9,IR)*(G*(1.-R)+CR(1,IR))
      H415=CR(9,IR)*(G*(1.-R)+CR(2,IR))
      H51=CR(9,IR)*((R-1.)*G*GSQ+GSQ*CR(3,IR)
     X+G*CR(5,IR)+CR(7,IR))
```

```
      H515=CR(9,IR)*((R-1.)*G*GSQ+GSQ*CR(4,IR)
     X+G*CR(6,IR)+CR(8,IR))
C     CALCULATION OF F4, F5
      EMGR=EXPF(G*(1.-R))
      F41=EMGR*(A41*EKR2+AB41*EKRP-H41)
      F415=EMGR*(A415*EKR2+AB415*EKRP-H415)
      F51=EMGR*(A51*EKR2+AB51*EKRP-H51)
      F515=EMGR*(A515*EKR2+AB515*EKRP-H515)
C     ETA DEPENDENCE
      IE=1
  511 E=ETA(IE)*ETA(IE)/(ETA(IE)*ETA(IE)+GSQ)
      B1=E*(2.*F41+E*F51)
      B15=E*(3.*F415+E*F515)
C     SUMMATIONS
      GOTO(519,520,521),JG
  519 BC1(IE,IR)=BC1(IE,IR)+4.*B1
      BC15(IE,IR)=BC15(IE,IR)+4.*B15
      GOTO509
  520 BC1(IE,IR)=BC1(IE,IR)+2.*B1
      BC15(IE,IR)=BC15(IE,IR)+2.*B15
      GOTO509
  521 BC1(IE,IR)=CST*(BC1(IE,IR)+B1)
      BC15(IE,IR)=CST*(BC15(IE,IR)+B15)
      GOTO509
C     OUTPUT
  515 DO522IR=1,NR
      WRITEOUTPUTTAPE6,581,RHO(IR),(ETA(IE),BC1(IE,IR),
     1BC15(IE,IR),IE=1,40)
  522 CONTINUE
  581 FORMAT(1H1F7.3/(F8.3,2E20.8))
C     G(1,ETA),G(1.5,ETA)FROM PROGRAM 8
C     G(XI) FOR ALPHA EQUALS 1.0
      G(1,1)=.20154125
      G(1,2)=.20993569
      G(1,3)=.21978752
      G(1,4)=.22626457
      G(1,5)=.21693650
      G(1,6)=.18155049
      G(1,7)=.13003021
      G(1,8)=.84137466E-1
      G(1,9)=.52481723E-1
      G(1,10)=.33041545E-1
      G(1,11)=.21327350E-1
      G(1,12)=.14224672E-1
      G(1,13)=.97755832E-2
      G(1,14)=.69195035E-2
      G(1,15)=.50199737E-2
      G(1,16)=.37282833E-2
      G(1,17)=.28227726E-2
      G(1,18)=.21768700E-2
      G(1,19)=.17043766E-2
      G(1,20)=.13537167E-2
      G(1,21)=.88540018E-3
      G(1,22)=.60372003E-3
      G(1,23)=.42581421E-3
      G(1,24)=.30909523E-3
      G(1,25)=.22985333E-3
      G(1,26)=.17456127E-3
      G(1,27)=.13498336E-3
      G(1,28)=.10606523E-3
      G(1,29)=.84515418E-4
      G(1,30)=.68194988E-4
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```
      G(1,31)=.45863959E-4
      G(1,32)=.31998922E-4
      G(1,33)=.23014008E-4
      G(1,34)=.16982160E-4
      G(1,35)=.12808580E-4
      G(1,36)=.98455451E-5
      G(1,37)=.76940523E-5
      G(1,38)=.61009600E-5
      G(1,39)=.49006505E-5
      G(1,40)=.39822131E-5
C     G(XI) FOR ALPHA EQUALS 1.5
      G(2,1)=.97895779E-1
      G(2,2)=.98713586E-1
      G(2,3)=.10012834
      G(2,4)=.99914737E-1
      G(2,5)=.92933235E-1
      G(2,6)=.75519875E-1
      G(2,7)=.52592502E-1
      G(2,8)=.33125947E-1
      G(2,9)=.20145591E-1
      G(2,10)=.12380441E-1
      G(2,11)=.78127421E-2
      G(2,12)=.51004198E-2
      G(2,13)=.34359130E-2
      G(2,14)=.23866579E-2
      G(2,15)=.17014134E-2
      G(2,16)=.12429380E-2
      G(2,17)=.92674737E-3
      G(2,18)=.70445783E-3
      G(2,19)=.54421783E-3
      G(2,20)=.42685848E-3
      G(2,21)=.27293737E-3
      G(2,22)=.18244641E-3
      G(2,23)=.12646843E-3
      G(2,24)=.90412962E-4
      G(2,25)=.66339907E-4
      G(2,26)=.49789584E-4
      G(2,27)=.38101513E-4
      G(2,28)=.29663067E-4
      G(2,29)=.23443080E-4
      G(2,30)=.18778225E-4
      G(2,31)=.12473916E-4
      G(2,32)=.86167022E-5
      G(2,33)=.61473403E-5
      G(2,34)=.45062138E-5
      G(2,35)=.33802395E-5
      G(2,36)=.25865050E-5
      G(2,37)=.20136259E-5
      G(2,38)=.15915941E-5
      G(2,39)=.12750054E-5
      G(2,40)=.10336716E-5
C     MULTIPLY BY WEIGHTING FACTORS
      DO730J=1,2
      DO720IE=1,19,2
  720 G(J,IE)=G(J,IE)/3.
      DO721IE=2,18,2
  721 G(J,IE)=G(J,IE)/6.
      G(J,20)=G(J,20)/4.
      DO722IE=21,29,2
  722 G(J,IE)=G(J,IE)/1.5
      DO723IE=22,28,2
  723 G(J,IE)=G(J,IE)/3.
```

```
      G(J,30)=0.5*G(J,30)
      DO724IE=31,39,2
  724 G(J,IE)=G(J,IE)/.75
      DO725IE=32,38,2
  725 G(J,IE)=G(J,IE)/1.5
  730 G(J,40)=G(J,40)/3.
C     INTEGRATION OVER G(ALPHA,ETA)
      WRITEOUTPUTTAPE6,1590
 1590 FORMAT(12H1PROGRAM 15./63H RHO, CONTRIBUTION TO TZZ STRESS FROM BC
     11, BC15, GAMMA (0, 100))
      DO1504IR=1,NR
      TZZ1=0.0
      TZZ15=0.0
      DO1505IE=1,40
      TZZ1=TZZ1+G(1,IE)*BC1(IE,IR)
 1505 TZZ15=TZZ15+G(2,IE)*BC15(IE,IR)
 1504 WRITEOUTPUTTAPE6,1591,RHO(IR),TZZ1,TZZ15
 1591 FORMAT(F12.3,2E20.8)
      CALLSYSTEM
      END  (   1 ,   1 ,   0 ,   1 ,   0   )                    ANL-30
```

```
C      PROGRAM 16. BOUNDARY CONDITION AND UZ
C      INTEGRALS OF G(ALPHA, ETA)XJ2(ETA,RHO)
C      FOR ETA (0,20)
C      BASED ON PROGRAM 6.
       DIMENSIONTABJ(120),TABY(120),TABS(120),TABX(120),RHO(15)
       DIMENSIONEJ2PR(40),EY2PR(40),E3J(40,15),G(2,40),ETA(40)
C      INPUT OF RHOS
       READINPUTTAPE7,680,NR,(RHO(I),I=1,NR)
  680  FORMAT(I2/(F12.3))
C      STORING OF ETAS
       DO1601IE=1,20
 1601  ETA(IE)=0.25*FLOATF(IE)
       DO1602IE=21,30
 1602  ETA(IE)=-5.0+0.5*FLOATF(IE)
       DO1603IE=31,40
 1603  ETA(IE)=IE-20
       DO1604IE=1,40
       E=ETA(IE)
       E2=E*E
       E4=E2*E2
C      CALCULATION OF EJ2PR,EY2PR
       Z1=BESJF(E,0.,1,120,XLOCF(TABJ))
       Z2=BESYF(E,0.,1,120,XLOCF(TABY))
       AJ0=TABJ(1)
       AJ1=TABJ(2)
       AY0=TABY(1)
       AY1=TABY(2)
       EJ2PR(IE)=(E-4./E)*AJ1+2.*AJ0
       EY2PR(IE)=(E-4./E)*AY1+2.*AY0
C      CALCULATION OF J2(ETA,RHO)
       DO602IR=1,NR
       R=RHO(IR)
       RE=R*E
       IF(20.-RE)604,605,605
  605  Z3=BESJF(RE,0.,1,120,XLOCF(TABS))
       Z4=BESYF(RE,0.,1,120,XLOCF(TABX))
       AJR0=TABS(1)
       AJR1=TABS(2)
       AYR0=TABX(1)
       AYR1=TABX(2)
       AJ2=2.*AJR1/RE-AJR0
       AY2=2.*AYR1/RE-AYR0
       EJ=EY2PR(IE)*AJ2-EJ2PR(IE)*AY2
       GOTO1605
  604  SN=SINF(E*(R-1.))
       CS=COSF(E*(R-1.))
       CRT=.63661977/SQRTF(R)
       R2=R*R
       R3=R*R2
       R4=R2*R2
       F12=((R-1.)/R2)*(-3.6328125*R+.8203125)
       F14=((R-1.)/R4)*(-.75942993*R3-4.0278625*R2
      X-1.0478210*R-.31723022)
       F16=((R-1.)/(R2*R4))*(1.7075908*R2*R3+3.41766*R4
      X+4.0406299*R3+3.5044026*R2+2.3519647*R+1.1276543)
       F21=(2.375*R-1.875)/R
       F23=(-1.7431641*R3+6.8115234*R2-1.9482422*R
```

```
      X-.30761719)/R3
       F25=(R4*(.91203690*R+1.4239311)+1.4299393*R3
      X+1.1175156*R2+.75342178*R+.51549912)/(R2*R3)
       F27=(R3*(-4.2488404*R4-3.2017329*R3-.74815527*R2
      X+.23361370*R-.55298433)-1.8727116*R2
      X-2.6781790*R-3.0809127)/(R3*R4)
       EJ=CRT*(CS*(1.+(F12+F14/E2+F16/E4)/E2)
      X+(SN/E)*(F21+(F23+F25/E2+F27/E4)/E2))
       GOTO1605
 1605  E3J(IE,IR)=E*E*EJ
  602  CONTINUE
 1604  CONTINUE
C      INTEGRATION OVER G
C      G(1,ETA),G(1.5,ETA)FROM PROGRAM 8
C      G(XI) FOR ALPHA EQUALS 1.0
       G(1,1)=.20154125
       G(1,2)=.20993569
       G(1,3)=.21978752
       G(1,4)=.22626457
       G(1,5)=.21693650
       G(1,6)=.18155049
       G(1,7)=.13003021
       G(1,8)=.84137466E-1
       G(1,9)=.52481723E-1
       G(1,10)=.33041545E-1
       G(1,11)=.21327350E-1
       G(1,12)=.14224672E-1
       G(1,13)=.97755832E-2
       G(1,14)=.69195035E-2
       G(1,15)=.50199737E-2
       G(1,16)=.37282833E-2
       G(1,17)=.28227726E-2
       G(1,18)=.21768700E-2
       G(1,19)=.17043766E-2
       G(1,20)=.13537167E-2
       G(1,21)=.88540018E-3
       G(1,22)=.60372003E-3
       G(1,23)=.42581421E-3
       G(1,24)=.30909523E-3
       G(1,25)=.22985333E-3
       G(1,26)=.17456127E-3
       G(1,27)=.13498336E-3
       G(1,28)=.10606523E-3
       G(1,29)=.84515418E-4
       G(1,30)=.68194988E-4
       G(1,31)=.45863959E-4
       G(1,32)=.31998922E-4
       G(1,33)=.23014008E-4
       G(1,34)=.16982160E-4
       G(1,35)=.12808580E-4
       G(1,36)=.98455451E-5
       G(1,37)=.76940523E-5
       G(1,38)=.61009600E-5
       G(1,39)=.49006505E-5
       G(1,40)=.39822131E-5
C      G(XI) FOR ALPHA EQUALS 1.5
       G(2,1)=.97895779E-1
       G(2,2)=.98713586E-1
       G(2,3)=.10012834
       G(2,4)=.99914737E-1
       G(2,5)=.92933235E-1
       G(2,6)=.75519875E-1
```

```
      G(2,7)=.52592502E-1
      G(2,8)=.33125947E-1
      G(2,9)=.20145591E-1
      G(2,10)=.12380441E-1
      G(2,11)=.78127421E-2
      G(2,12)=.51004198E-2
      G(2,13)=.34359130E-2
      G(2,14)=.23866579E-2
      G(2,15)=.17014134E-2
      G(2,16)=.12429380E-2
      G(2,17)=.92674737E-3
      G(2,18)=.70445783E-3
      G(2,19)=.54421783E-3
      G(2,20)=.42685848E-3
      G(2,21)=.27293737E-3
      G(2,22)=.18244641E-3
      G(2,23)=.12646843E-3
      G(2,24)=.90412962E-4
      G(2,25)=.66339907E-4
      G(2,26)=.49789584E-4
      G(2,27)=.38101513E-4
      G(2,28)=.29663067E-4
      G(2,29)=.23443080E-4
      G(2,30)=.18778225E-4
      G(2,31)=.12473916E-4
      G(2,32)=.86167022E-5
      G(2,33)=.61473403E-5
      G(2,34)=.45062138E-5
      G(2,35)=.33802395E-5
      G(2,36)=.25865050E-5
      G(2,37)=.20136259E-5
      G(2,38)=.15915941E-5
      G(2,39)=.12750054E-5
      G(2,40)=.10336716E-5
C     MULTIPLY BY WEIGHTING FACTORS
      DO730J=1,2
      DO720IE=1,19,2
  720 G(J,IE)=G(J,IE)/3.
      DO721IE=2,18,2
  721 G(J,IE)=G(J,IE)/6.
      G(J,20)=G(J,20)/4.
      DO722IE=21,29,2
  722 G(J,IE)=G(J,IE)/1.5
      DO723IE=22,28,2
  723 G(J,IE)=G(J,IE)/3.
      G(J,30)=0.5*G(J,30)
      DO724IE=31,39,2
  724 G(J,IE)=G(J,IE)/.75
      DO725IE=32,38,2
  725 G(J,IE)=G(J,IE)/1.5
  730 G(J,40)=G(J,40)/3.
      WRITEOUTPUTTAPE6,1680
 1680 FORMAT(4H1RHO/50H CONTRIBUTION OF J2 TERM TO TZZ1, TZZ15, UZ1, UZ1
     15)
      DO1606IR=1,NR
      TZZ1=0.0
      TZZ15=0.0
      UZ1=0.0
      UZ15=0.0
      DO1607IE=1,40
      TZZ1=TZZ1+G(1,IE)*E3J(IE,IR)*ETA(IE)
      TZZ15=TZZ15+G(2,IE)*E3J(IE,IR)*ETA(IE)
```

```
      UZ1=UZ1-G(1,IE)*E3J(IE,IR)
 1607 UZ15=UZ15-1.5*G(2,IE)*E3J(IE,IR)
      WRITEOUTPUTTAPE6,1681,RHO(IR),TZZ1,TZZ15,UZ1,UZ15
 1681 FORMAT(F12.3/4E20.8)
 1606 CONTINUE
      CALLSYSTEM
      END  (   1 ,    1 ,    0 ,    1 ,    0   )                    ANL-30
```

```
C     PROGRAM 9. BOUNDARY CONDITION
C     INTEGRALS OF G(ALPHA, ETA) X BC(ETA,RHO)
C     BC ARE INTEGRALS OVER LARGE GAMMA  ≥10
      DIMENSIONRHO(15),ETA(40),BC1(40,15),BC15(40,15),CR(9,15)
      DIMENSIONG(2,40)
      DIMENSIONTABR(66),TABG(66)
      CST3=.135094912
C     INPUT OF RHOS
C     RHOS TO BE MONOTONICALLY INCREASING
      READINPUTTAPE7,580,NR,(RHO(IR),IR=1,NR)
  580 FORMAT(I2/(F12.3))
C     STORING OF ETAS
      DO1501IE=1,20
 1501 ETA(IE)=0.25*FLOATF(IE)
      DO1502IE=21,30
 1502 ETA(IE)=-5.0+0.5*FLOATF(IE)
      DO1503IE=31,40
 1503 ETA(IE)=IE-20
C     FUNCTIONS OF RHO
C     CR FUNCTIONS FROM PROGRAM 5
      DO503IR=1,NR
      R=RHO(IR)
      CR(9,IR)=1./SQRTF(R)
      R1=(R-1.)/R
      R2=R1*(R+1.)/R
      R3=R1*(R*R+R+1.)/(R*R)
      AL1=-1.875*R1
      AL2=-.8203125*R2+3.515625*R1
      AL3=.30761719*R3+1.5380859*R2-5.0537109*R1
      ALS1=-2.375*R1
      ALS2=-3.6328125*R2+5.640625*R1
      ALS3=-1.7431641*R3+8.6279279*R2-4.7685547*R1
      CR(1,IR)=AL1+R*(1.-ALS1)
      CR(2,IR)=-0.5+AL1+R*(1.5-ALS1)
      CR(3,IR)=-AL1-1.+R*ALS1
      CR(4,IR)=-AL1-0.5+R*(ALS1-0.5)
      CR(5,IR)=-AL2-AL1+4.875*(R-1.)+R*ALS2
      CR(6,IR)=-AL2-0.5*AL1-4.625+R*(ALS2-0.5*ALS1+5.125)
      CR(7,IR)=-AL3-AL2-4.875*AL1+R*(ALS3+4.875*ALS1-2.5)
      CR(8,IR)=-AL3-0.5*AL2-4.625*AL1+0.75+R*(ALS3
     X-0.5*ALS2+5.125*ALS1-1.5)
  503 CONTINUE
C     GAMMA ROUTING
      JG=1
      IG=20
      GOTO903
  909 JG=2
      IG=21
      GOTO903
  911 IG=IG+2
      IF(500-IG)912,913,903
  912 JG=3
      IG=22
      GOTO903
  913 JG=4
      GOTO903
C     ETA ROUTING
```

```
  905 IE=IE+1
      IF(40-IE)906,511,511
C     RHO ROUTING
  906 IR=IR+1
      IF(NR-IR)908,514,514
  908 GOTO(909,911,911,915),JG
C     FUNCTIONS OF GAMMA IN INTEGRANDS
  903 G=0.5*FLOATF(IG)
      GSQ=G*G
      G3=G*GSQ
      G4=GSQ*GSQ
C     SUBROUTINES FOR AK, EK2, DELTA1, DELTA2,
C     ASYMPTOTIC EXPANSIONS
      AK=-G-0.5-1.875/G+1.875/GSQ-1.0546875/G3
     1-1.40625/G4+(7.2509766/G-21.09375/GSQ+58.152008/G3)/G4
      EK2=(SQRTF(1.5707963/G))*(1.+1.875/G+.8203125/GSQ
     1-.30761719/G3+.31723022/G4+(-.51549912/G
     2+1.1276543/GSQ-3.0809127/G3)/G4)
      DELTA1=-GSQ*G3-2.*G4-7.125*G3-3.75*GSQ+4.2890625*G
     1-16.875+2.0654297/G+104.0625/GSQ
      DELTA2=-G4-3.*G3-3.*GSQ+1.125*G+6.6796875/G-19.6875/GSQ
      DEK1=EK2*(DELTA1+DELTA2)
      DEK15=EK2*(DELTA1+1.5*DELTA2)
      FG=GSQ*(GSQ*GSQ+8.*GSQ+12.-2.*GSQ*AK)
      HG=GSQ*(GSQ-(GSQ+3.)*AK)
      A41=GSQ*(-GSQ*GSQ-9.*GSQ-12.+(3.*GSQ+3.)*AK)/DEK1
      A415=GSQ*(-GSQ*GSQ-9.*GSQ-12.+(3.*GSQ+1.5)*AK)/DEK15
      AB41=-HG/DEK1
      AB415=-HG/DEK15
      A51=((GSQ+4.-2.*AK)*FG-4.*HG)/DEK1
      A515=((GSQ+4.-2.*AK)*FG-6.*HG)/DEK15
      AB51=-AK*FG/DEK1
      AB515=-AK*FG/DEK15
C     FUNCTIONS OF GAMMA, RHO IN INTEGRAND
      IR=1
  514 R=RHO(IR)
      GD=G*(R-1.)
      IF(25.-GD)908,907,907
  907 GR=G*R
      IF(10.-GR)516,517,517
  517 Z2=BESKF(GR,0.,1,66,XLOCF(TABG))
      AKR0=TABG(1)
      AKR1=TABG(2)
      EKR2=AKR0+2.*AKR1/GR
      EKRP=-(2.*AKR0+(GR+4./GR)*AKR1)
      GOTO518
C     ASYMPTOTIC EXPANSIONS FOR EKR2, EKRP
  516 RT1=SQRTF(1.5707963/GR)
      RT2=-SQRTF(1.5707963*GR)
      GR2=GR*GR
      GR3=GR*GR2
      GR4=GR2*GR2
      EKR2=RT1*(1.+1.875/GR+.8203125/GR2-.30761719/GR3
     X+.31723022/GR4-.51549912/(GR2*GR3)
     X+1.1276543/(GR3*GR3)-3.0809127/(GR4*GR3))
      EKRP=RT2*(1.+2.375/GR+3.6328125/GR2+1.7431641/GR3
     X-.75942993/GR4+.91203690/(GR2*GR3)
     X-1.7075908/(GR3*GR3)+4.2488404/(GR3*GR4))
      GOTO518
C     CALCULATION OF H4, H5
  518 H41=CR(9,IR)*(G*(1.-R)+CR(1,IR))
      H415=CR(9,IR)*(G*(1.-R)+CR(2,IR))
```

```
      H51=CR(9,IR)*((R-1.)*G*GSQ+GSQ*CR(3,IR)
     X+G*CR(5,IR)+CR(7,IR))
      H515=CR(9,IR)*((R-1.)*G*GSQ+GSQ*CR(4,IR)
     X+G*CR(6,IR)+CR(8,IR))
C     CALCULATION OF F4, F5
      EMGR=EXPF(G*(1.-R))
      F41=EMGR*(A41*EKR2+AB41*EKRP-H41)
      F415=EMGR*(A415*EKR2+AB415*EKRP-H415)
      F51=EMGR*(A51*EKR2+AB51*EKRP-H51)
      F515=EMGR*(A515*EKR2+AB515*EKRP-H515)
C     ETA DEPENDENCE
      IE=1
  511 E=ETA(IE)*ETA(IE)/(ETA(IE)*ETA(IE)+GSQ)
      B1=E*(2.*F41+E*F51)
      B15=E*(3.*F415+E*F515)
C     SUMMATIONS
      GOTO(904,910,914,914),JG
  904 BC1(IE,IR)=CST3*B1
      BC15(IE,IR)=CST3*B15
      GOTO905
  910 BC1(IE,IR)=BC1(IE,IR)+4.*CST3*B1
      BC15(IE,IR)=BC15(IE,IR)+4.*CST3*B15
      GOTO905
  914 BC1(IE,IR)=BC1(IE,IR)+2.*CST3*B1
      BC15(IE,IR)=BC15(IE,IR)+2.*CST3*B15
      GOTO905
C     OUTPUT
  915 DO916IR=1,NR
      WRITEOUTPUTTAPE6,980,RHO(IR),(ETA(IE),BC1(IE,IR),
     1BC15(IE,IR),IE=1,40)
  980 FORMAT(1H1F8.3/(F8.3,2E20.8))
  916 CONTINUE
C     INTEGRATION OVER G
C     G(1,ETA),G(1.5,ETA)FROM PROGRAM 8
C     G(XI) FOR ALPHA EQUALS 1.0
      G(1,1)=.20154125
      G(1,2)=.20993569
      G(1,3)=.21978752
      G(1,4)=.22626457
      G(1,5)=.21693650
      G(1,6)=.18155049
      G(1,7)=.13003021
      G(1,8)=.84137466E-1
      G(1,9)=.52481723E-1
      G(1,10)=.33041545E-1
      G(1,11)=.21327350E-1
      G(1,12)=.14224672E-1
      G(1,13)=.97755832E-2
      G(1,14)=.69195035E-2
      G(1,15)=.50199737E-2
      G(1,16)=.37282833E-2
      G(1,17)=.28227726E-2
      G(1,18)=.21768700E-2
      G(1,19)=.17043766E-2
      G(1,20)=.13537167E-2
      G(1,21)=.88540018E-3
      G(1,22)=.60372003E-3
      G(1,23)=.42581421E-3
      G(1,24)=.30909523E-3
      G(1,25)=.22985333E-3
      G(1,26)=.17456127E-3
      G(1,27)=.13498336E-3
```

```
      G(1,28)=.10606523E-3
      G(1,29)=.84515418E-4
      G(1,30)=.68194988E-4
      G(1,31)=.45863959E-4
      G(1,32)=.31998922E-4
      G(1,33)=.23014008E-4
      G(1,34)=.16982160E-4
      G(1,35)=.12808580E-4
      G(1,36)=.98455451E-5
      G(1,37)=.76940523E-5
      G(1,38)=.61009600E-5
      G(1,39)=.49006505E-5
      G(1,40)=.39822131E-5
C     G(XI) FOR ALPHA EQUALS 1.5
      G(2,1)=.97895779E-1
      G(2,2)=.98713586E-1
      G(2,3)=.10012834
      G(2,4)=.99914737E-1
      G(2,5)=.92933235E-1
      G(2,6)=.75519875E-1
      G(2,7)=.52592502E-1
      G(2,8)=.33125947E-1
      G(2,9)=.20145591E-1
      G(2,10)=.12380441E-1
      G(2,11)=.78127421E-2
      G(2,12)=.51004198E-2
      G(2,13)=.34359130E-2
      G(2,14)=.23866579E-2
      G(2,15)=.17014134E-2
      G(2,16)=.12429380E-2
      G(2,17)=.92674737E-3
      G(2,18)=.70445783E-3
      G(2,19)=.54421783E-3
      G(2,20)=.42685848E-3
      G(2,21)=.27293737E-3
      G(2,22)=.18244641E-3
      G(2,23)=.12646843E-3
      G(2,24)=.90412962E-4
      G(2,25)=.66339907E-4
      G(2,26)=.49789584E-4
      G(2,27)=.38101513E-4
      G(2,28)=.29663067E-4
      G(2,29)=.23443080E-4
      G(2,30)=.18778225E-4
      G(2,31)=.12473916E-4
      G(2,32)=.86167022E-5
      G(2,33)=.61473403E-5
      G(2,34)=.45062138E-5
      G(2,35)=.33802395E-5
      G(2,36)=.25865050E-5
      G(2,37)=.20136259E-5
      G(2,38)=.15915941E-5
      G(2,39)=.12750054E-5
      G(2,40)=.10336716E-5
C     MULTIPLY BY WEIGHTING FACTORS
      DO730J=1,2
      DO720IE=1,19,2
  720 G(J,IE)=G(J,IE)/3.
      DO721IE=2,18,2
  721 G(J,IE)=G(J,IE)/6.
      G(J,20)=G(J,20)/4.
      DO722IE=21,29,2
```

```
722 G(J,IE)=G(J,IE)/1.5
    DO723IE=22,28,2
723 G(J,IE)=G(J,IE)/3.
    G(J,30)=0.5*G(J,30)
    DO724IE=31,39,2
724 G(J,IE)=G(J,IE)/.75
    DO725IE=32,38,2
725 G(J,IE)=G(J,IE)/1.5
730 G(J,40)=G(J,40)/3.
    WRITEOUTPUTTAPE6,981
981 FORMAT(11H1PROGRAM 9./70H RHO, CONTRIBUTION TO TZZ STRESS FROM BC1
   1, BC15, GAMMA GREATER THAN 10)
    DO920IR=1,NR
    TZZ1=0.0
    TZZ15=0.0
    DO921IE=1,40
    TZZ1=TZZ1+G(1,IE)*BC1(IE,IR)
921 TZZ15=TZZ15+G(2,IE)*BC15(IE,IR)
920 WRITEOUTPUTTAPE6,982,RHO(IR),TZZ1,TZZ15
982 FORMAT(F12.3,2E20.8)
    CALLSYSTEM
    END  (  1 ,   1 ,   0 ,   1 ,   0  )                        ANL-30
```

```
C      PROGRAM 10A. BOUNDARY CONDITION.
C      INTEGRALS OF G(ALPHA,ETA)X W(ETA,RHC)
C      BASED ON PROGRAM 10.
C      CALCULATION OF FUNCTIONS OF SINE AND COSINE
C      INTEGRALS BASED ON ARCSI1
C      REVISED TO USE ASYMPTOTIC EXPANSIONS FOR X
C      GREATER THAN 11
       DIMENSIONRHO(15),ETA(40),CR(9,15),W1(40),W15(40),CF(6)
       DIMENSIONA0(4),A2(4),A4(4),A6(4),QP(4)
       DIMENSIONG(2,40),WR1(40,15),WR15(40,15)
C      INPUT OF RHOS
       READINPUTTAPE7,1080,NR,(RHO(IR),IR=1,NR)
       PI=3.14159265
C      STORING OF ETAS
       DO1001IE=1,20
 1001  ETA(IE)=0.25*FLOATF(IE)
       DO1002IE=21,30
 1002  ETA(IE)=-5.+0.5*FLOATF(IE)
       DO1003IE=31,40
 1003  ETA(IE)=IE-20
       CST8=.81056947
C      CR FUNCTIONS FROM PROGRAM 5
       DO503IR=1,NR
       R=RHO(IR)
       CR(9,IR)=1./SQRTF(R)
       R1=(R-1.)/R
       R2=R1*(R+1.)/R
       R3=R1*(R*R+R+1.)/(R*R)
       AL1=-1.875*R1
       AL2=-.8203125*R2+3.515625*R1
       AL3=.30761719*R3+1.5380859*R2-5.0537109*R1
       ALS1=-2.375*R1
       ALS2=-3.6328125*R2+5.640625*R1
       ALS3=-1.7431641*R3+8.6279279*R2-4.7685547*R1
       CR(1,IR)=AL1+R*(1.-ALS1)
       CR(2,IR)=-0.5+AL1+R*(1.5-ALS1)
       CR(3,IR)=-AL1-1.+R*ALS1
       CR(4,IR)=-AL1-0.5+R*(ALS1-0.5)
       CR(5,IR)=-AL2-AL1+4.875*(R-1.)+R*ALS2
       CR(6,IR)=-AL2-0.5*AL1-4.625+R*(ALS2-0.5*ALS1+5.125)
       CR(7,IR)=-AL3-AL2-4.875*AL1+R*(ALS3+4.875*ALS1-2.5)
       CR(8,IR)=-AL3-0.5*AL2-4.625*AL1+0.75+R*(ALS3
      X-0.5*ALS2+5.125*ALS1-1.5)
  503  CONTINUE
       IR=1
 1055  WRITE OUTPUT TAPE6,1081,RHO(IR)
       IF(RHO(IR)-1.)1071,1071,1072
 1072  D=RHO(IR)-1.
       IE=1
 1053  ED=D*ETA(IE)
       ED2=ED*ED
C      SUBROUTINE FOR FUNCTIONS OF SINE AND COSINE INTEGRALS,
C      FUNCTIONS C1 THROUGH C6.
 1004  IF(ED-11.)1023,1005,1005
C      SUBROUTINE FOR ARGUMENT GREATER THAN 11 USING
C      ASYMPTOTIC EXPANSIONS
 1005  DO1006J=1,6
```

```
      CFJ=1.0
      CNJ=1.0
      DO1007N=1,15
      XN=N
      GOTO(1011,1012,1013,1014,1015,1016),J
 1011 XNJ=2*N*(2*N-1)
      GOTO1008
 1012 XNJ=2*N*(2*N+1)
      GOTO1008
 1013 XNJ=2*(N+1)*(2*N+1)
      GOTO1008
 1014 XNJ=2*(N+1)*(2N-1)
      GOTO1008
 1015 XNJ=2.*(XN+1.)*(XN+1.)*(2.*XN+1.)/XN
      GOTO1008
 1016 XNJ=2.*(XN+1.)*(XN+1.)*(2.*XN+3.)/XN
 1008 CNJ=-CNJ*XNJ/ED2
      IF(XNJ-ED2)1009,1010,1010
 1009 CFJ=CFJ+CNJ
 1007 FCNJ=CNJ
      GOTO1006
 1010 CFJ=CFJ-0.5*FCNJ
 1006 CF(J)=CFJ
      GOTO1050
 1023 IF(ED-1.0)1024,1025,1025
C     SUBROUTINE FOR ARG. BETWEEN 0 AND 1
 1024 M=1
      MM=1
      P=ED
      PP=ED2
      PPP=ED2*ED2
      SI=ED*(1.0-PP/18.0)
 1026 Q=0.5*PP
      CI=0.57721567+LOGF(ED)+0.25*PP*(-1.0+PP/24.0)
 1027 ERROR=1.0E-15
 1028 DO1029N=2,50
      AN=N
      BN=2.0*AN
      CN=2.0*BN
 1030 IF(M)1031,1032,1032
 1032 P=P*PPP/((CN-3.0)*(CN-4.0)*(CN-5.0)*(CN-6.0))
      TERM=P*(1.0/(CN-3.0)-PP/(((CN-1.0)**2)*(CN-2.0)))
      IF(ABSF(TERM)-ERROR)1033,1033,1034
 1034 SI=SI+TERM
      GOTO1031
 1033 M=-1
 1031 IF(MM)1035,1036,1036
 1036 Q=Q*PPP/((CN-2.0)*(CN-3.0)*(CN-4.0)*(CN-5.0))
      TERM=-Q*(1.0/(CN-2.0)-PP/((CN-1.0)*CN**2))
      IF(ABSF(TERM)-ERROR)1037,1037,1038
 1038 CI=CI+TERM
      GOTO1035
 1037 MM=-1
 1035 IF(M)1040,1029,1029
 1029 CONTINUE
 1040 SSI=SI-1.57079633
      SNF=SINF(ED)
      CSF=COSF(ED)
      CF(1)=ED*(SNF*CI-CSF*SSI)
      CF(2)=-ED2*(CSF*CI+SNF*SSI)
      CF(3)=0.5*ED2*(1.0-CF(1))
      CF(4)=0.5*(CF(1)+CF(2))
```

```
      CF(5)=0.25*ED2*(CF(1)-CF(2))
      CF(6)=ED2*(CF(2)-CF(3))/6.0
      GOTO1050
C     SUBROUTINE FOR ARG BETWEEN 1 AND 18
 1025 A0(1)=38.102495
      A2(1)=335.67732
      A4(1)=265.187033
      A6(1)=38.027264
      A0(2)=157.105423
      A2(2)=570.236280
      A4(2)=322.624911
      A6(2)=40.021433
      A0(3)=21.821899
      A2(3)=352.018498
      A4(3)=302.757865
      A6(3)=42.242855
      A0(4)=449.690326
      A2(4)=1114.978885
      A4(4)=482.485984
      A6(4)=48.196927
 1045 REC=1.0/ED2
 1046 DO1047I=1,4
 1047 QP(I)=REC*(REC*(REC*A0(I)+A2(I))+A4(I))+A6(I)+ED2
 1048 CF(1)=QP(1)/QP(2)
      CF(2)=QP(3)/QP(4)
      CF(3)=0.5*ED2*(1.0-CF(1))
      CF(4)=0.5*(CF(1)+CF(2))
      CF(5)=0.25*ED2*(CF(1)-CF(2))
      CF(6)=ED2*(CF(2)-CF(3))/6.0
      GOTO1050
C     OUTPUT OF CF(J)
 1050 WRITEOUTPUTTAPE6,1082,ETA(IE),(CF(J),J=1,6)
C     CALCULATION OF W FOR ALPHA EQUALS 1.0,1.5
      W1(IE)=CST8*CR(9,IR)*(-2.*CF(2)+2.*CR(1,IR)*CF(1)
     1+(6.*CF(6)+2.*CR(3,IR)*CF(5))/(D*D)+CR(5,IR)*CF(3)/D
     2+CR(7,IR)*CF(4))/D
      W15(IE)=CST8*CR(9,IR)*(-3.*CF(2)+3.*CR(2,IR)*CF(1)
     1+(6.*CF(6)+2.*CR(4,IR)*CF(5))/(D*D)+CR(6,IR)*CF(3)/D
     2+CR(8,IR)*CF(4))/D
      WR1(IE,IR)=W1(IE)
      WR15(IE,IR)=W15(IE)
C     ROUTING AND OUTPUT OF W1, W15
 1051 IE=IE+1
      IF(40-IE)1052,1053,1053
 1052 WRITEOUTPUTTAPE6,1083,RHO(IR),(ETA(IE),W1(IE),
     1W15(IE),IE=1,40)
      IR=IR+1
      IF(NR-IR)1054,1055,1055
C     W1, W15 FOR RHO EQUALS 1.0
 1071 DO1073IE=1,40
      E=ETA(IE)
      W1(IE)=CST8*(PI*(CR(1,IR)+0.25*CR(3,IR)*E*E
     1+0.25*CR(7,IR))+0.5*E*CR(5,IR))*E
      W15(IE)=CST8*(PI*(1.5*CR(2,IR)+0.25*CR(4,IR)*E*E
     1+0.25*CR(8,IR))+0.5*E*CR(6,IR))*E
      WR1(IE,IR)=W1(IE)
      WR15(IE,IR)=W15(IE)
 1073 WRITEOUTPUTTAPE6,1084,E,W1(IE),W15(IE)
      IR=IR+1
      IF(NR-IR)1054,1055,1055
 1054 CONTINUE
 1080 FORMAT(I2/(F12.3))
```

```
 1081 FORMAT(1H1F8.3)
 1082 FORMAT(F8.3,6E17.8)
 1083 FORMAT(1H1F8.3/(F8.3,2E20.8))
 1084 FORMAT(F8.3,2E20.8)
C       INTEGRATION OVER G
C       G(1,ETA),G(1.5,ETA)FROM PROGRAM 8
C       G(XI) FOR ALPHA EQUALS 1.0
        G(1,1)=.20154125
        G(1,2)=.20993569
        G(1,3)=.21978752
        G(1,4)=.22626457
        G(1,5)=.21693650
        G(1,6)=.18155049
        G(1,7)=.13003021
        G(1,8)=.84137466E-1
        G(1,9)=.52481723E-1
        G(1,10)=.33041545E-1
        G(1,11)=.21327350E-1
        G(1,12)=.14224672E-1
        G(1,13)=.97755832E-2
        G(1,14)=.69195035E-2
        G(1,15)=.50199737E-2
        G(1,16)=.37282833E-2
        G(1,17)=.28227726E-2
        G(1,18)=.21768700E-2
        G(1,19)=.17043766E-2
        G(1,20)=.13537167E-2
        G(1,21)=.88540018E-3
        G(1,22)=.60372003E-3
        G(1,23)=.42581421E-3
        G(1,24)=.30909523E-3
        G(1,25)=.22985333E-3
        G(1,26)=.17456127E-3
        G(1,27)=.13498336E-3
        G(1,28)=.10606523E-3
        G(1,29)=.84515418E-4
        G(1,30)=.68194988E-4
        G(1,31)=.45863959E-4
        G(1,32)=.31998922E-4
        G(1,33)=.23014008E-4
        G(1,34)=.16982160E-4
        G(1,35)=.12808580E-4
        G(1,36)=.98455451E-5
        G(1,37)=.76940523E-5
        G(1,38)=.61009600E-5
        G(1,39)=.49006505E-5
        G(1,40)=.39822131E-5
C       G(XI) FOR ALPHA EQUALS 1.5
        G(2,1)=.97895779E-1
        G(2,2)=.98713586E-1
        G(2,3)=.10012834
        G(2,4)=.99914737E-1
        G(2,5)=.92933235E-1
        G(2,6)=.75519875E-1
        G(2,7)=.52592502E-1
        G(2,8)=.33125947E-1
        G(2,9)=.20145591E-1
        G(2,10)=.12380441E-1
        G(2,11)=.78127421E-2
        G(2,12)=.51004198E-2
        G(2,13)=.34359130E-2
        G(2,14)=.23866579E-2
```

```
      G(2,15)=.17014134E-2
      G(2,16)=.12429380E-2
      G(2,17)=.92674737E-3
      G(2,18)=.70445783E-3
      G(2,19)=.54421783E-3
      G(2,20)=.42685848E-3
      G(2,21)=.27293737E-3
      G(2,22)=.18244641E-3
      G(2,23)=.12646843E-3
      G(2,24)=.90412962E-4
      G(2,25)=.66339907E-4
      G(2,26)=.49789584E-4
      G(2,27)=.38101513E-4
      G(2,28)=.29663067E-4
      G(2,29)=.23443080E-4
      G(2,30)=.18778225E-4
      G(2,31)=.12473916E-4
      G(2,32)=.86167022E-5
      G(2,33)=.61473403E-5
      G(2,34)=.45062138E-5
      G(2,35)=.33802395E-5
      G(2,36)=.25865050E-5
      G(2,37)=.20136259E-5
      G(2,38)=.15915941E-5
      G(2,39)=.12750054E-5
      G(2,40)=.10336716E-5
C     MULTIPLY BY WEIGHTING FACTORS
      DO730J=1,2
      DO720IE=1,19,2
  720 G(J,IE)=G(J,IE)/3.
      DO721IE=2,18,2
  721 G(J,IE)=G(J,IE)/6.
      G(J,20)=G(J,20)/4.
      DO722IE=21,29,2
  722 G(J,IE)=G(J,IE)/1.5
      DO723IE=22,28,2
  723 G(J,IE)=G(J,IE)/3.
      G(J,30)=0.5*G(J,30)
      DO724IE=31,39,2
  724 G(J,IE)=G(J,IE)/.75
      DO725IE=32,38,2
  725 G(J,IE)=G(J,IE)/1.5
  730 G(J,40)=G(J,40)/3.
      WRITEOUTPUTTAPE6,1085
 1085 FORMAT(13H1PROGRAM 10A./56H RHO, CONTRIBUTION TO TZZ STRESS FROM W
     1(ETA, RHO, ALPHA))
      DO1060IR=1,NR
      TZZ1=0.0
      TZZ15=0.0
      DO1061IE=1,40
      TZZ1=TZZ1+G(1,IE)*WR1(IE,IR)
 1061 TZZ15=TZZ15+G(2,IE)*WR15(IE,IR)
 1060 WRITEOUTPUTTAPE6,1084,RHO(IR),TZZ1,TZZ15
      CALLSYSTEM
      END  (  1 ,  1 ,  0 ,  1 ,  0  )                              ANL-30
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```
C     PROGRAM 14. BOUNDARY CONDITION (AND UZ).
C     INTEGRALS OF G(ALPHA, ETA) X J2(ETA, RHO) AND
C     G(ALPHA, ETA) X W(ETA, RHO) FOR ETA (20, 100, 5)
C     AND SMALL RHO.
      DIMENSIONETA(17),RHO(15),E3J(17,15),GL(2,17),
     1WGL(2,17),CR(9,15),W1(17),W15(17),WR1(17,15),WR15(17,15),
     2A0(4),A2(4),A4(4),A6(4),QP(4),CF(6)
C     INPUT OF RHOS
      READINPUTTAPE7,680,NR,(RHO(I),I=1,NR)
  680 FORMAT(I2/(F12.3))
C     STORING OF ETAS
      DO1401IE=1,17
 1401 ETA(IE)=15+5*IE
C     CALCULATION OF J2(ETA,RHO)
      DO1402IR=1,NR
      R=RHO(IR)
      WRITEOUTPUTTAPE6,683,R
  683 FORMAT(1H1F8.3)
      CRT=.63661977/SQRTF(R)
      R2=R*R
      R3=R*R2
      R4=R2*R2
      F12=((R-1.)/R2)*(-3.6328125*R+.8203125)
      F14=((R-1.)/R4)*(-.75942993*R3-4.0278625*R2
     X-1.0478210*R-.31723022)
      F16=((R-1.)/(R2*R4))*(1.7075908*R2*R3+3.41766*R4
     X+4.0406299*R3+3.5044026*R2+2.3519647*R+1.1276543)
      F21=(2.375*R-1.875)/R
      F23=(-1.7431641*R3+6.8115234*R2-1.9482422*R
     X-.30761719)/R3
      F25=(R4*(.91203690*R+1.4239311)+1.4299393*R3
     X+1.1175156*R2+.75342178*R+.51549912)/(R2*R3)
      F27=(R3*(-4.2488404*R4-3.2017329*R3-.74815527*R2
     X+.23361370*R-.55298433)-1.8727116*R2
     X-2.6781790*R-3.0809127)/(R3*R4)
      DO1403IE=1,17
      E=ETA(IE)
      E2=E*E
      E4=E2*E2
      RE=R*E
  604 SN=SINF(E*(R-1.))
      CS=COSF(E*(R-1.))
      EJ=CRT*(CS*(1.+(F12+F14/E2+F16/E4)/E2)
     X+(SN/E)*(F21+(F23+F25/E2+F27/E4)/E2))
 1605 E3J(IE,IR)=E*E*EJ
      E4J=E*E3J(IE,IR)
 1403 WRITEOUTPUTTAPE6,1480,E,EJ,E3J(IE,IR),E4J
 1402 CONTINUE
      PI=3.14159265
      CST8=.81056947
C     CR FUNCTIONS FROM PROGRAM 5
      DO503IR=1,NR
      R=RHO(IR)
      CR(9,IR)=1./SQRTF(R)
      R1=(R-1.)/R
      R2=R1*(R+1.)/R
      R3=R1*(R*R+R+1.)/(R*R)
```

```
      AL1=-1.875*R1
      AL2=-.8203125*R2+3.515625*R1
      AL3=.30761719*R3+1.5380859*R2-5.0537109*R1
      ALS1=-2.375*R1
      ALS2=-3.6328125*R2+5.640625*R1
      ALS3=-1.7431641*R3+8.6279279*R2-4.7685547*R1
      CR(1,IR)=AL1+R*(1.-ALS1)
      CR(2,IR)=-0.5+AL1+R*(1.5-ALS1)
      CR(3,IR)=-AL1-1.+R*ALS1
      CR(4,IR)=-AL1-0.5+R*(ALS1-0.5)
      CR(5,IR)=-AL2-AL1+4.875*(R-1.)+R*ALS2
      CR(6,IR)=-AL2-0.5*AL1-4.625+R*(ALS2-0.5*ALS1+5.125)
      CR(7,IR)=-AL3-AL2-4.875*AL1+R*(ALS3+4.875*ALS1-2.5)
      CR(8,IR)=-AL3-0.5*AL2-4.625*AL1+0.75+R*(ALS3
     X-C.5*ALS2+5.125*ALS1-1.5)
  503 CCNTINUE
      IR=1
 1055 WRITE OUTPUT TAPE6,1081,RHO(IR)
      IF(RHC(IR)-1.)1071,1071,1072
 1072 D=RHO(IR)-1.
      IE=1
 1053 EC=C*ETA(IE)
      EC2=EC*ED
C     SUBROUTINE FOR FUNCTIONS CF SINE AND COSINE INTEGRALS,
C     FUNCTIONS C1 THROUGH C6.
 1004 IF(ED-11.)1023,1005,1005
C     SUBROUTINE FOR ARGUMENT GREATER THAN 11 USING
C     ASYMPTOTIC EXPANSIONS
 1005 DC1006J=1,6
      CFJ=1.0
      CNJ=1.0
      DC1007N=1,15
      XN=N
      GCTO(1011,1012,1013,1014,1015,1016),J
 1011 XNJ=2*N*(2*N-1)
      GCTO1008
 1012 XNJ=2*N*(2*N+1)
      GCTO1008
 1013 XNJ=2*(N+1)*(2*N+1)
      GCTO1008
 1014 XNJ=2*(N+1)*(2N-1)
      GCTO1008
 1015 XNJ=2.*(XN+1.)*(XN+1.)*(2.*XN+1.)/XN
      GCTO1008
 1016 XNJ=2.*(XN+1.)*(XN+1.)*(2.*XN+3.)/XN
 1008 CNJ=-CNJ*XNJ/ED2
      IF(XNJ-ED2)1009,1010,1010
 1009 CFJ=CFJ+CNJ
 1007 FCNJ=CNJ
      GCTO1006
 1010 CFJ=CFJ-0.5*FCNJ
 1006 CF(J)=CFJ
      GCTO1050
 1023 IF(ED-1.0)1024,1025,1025
C     SUBROUTINE FOR ARG. BETWEEN 0 AND 1
 1024 M=1
      MM=1
      P=EC
      PP=ED2
      PPP=EC2*EC2
      SI=EC*(1.0-PP/18.0)
 1026 Q=0.5*PP
```

```
      CI=0.57721567+LOGF(ED)+0.25*PP*(-1.0+PP/24.0)
 1027 ERROR=1.0E-15
 1028 DO1029N=2,50
      AN=N
      BN=2.0*AN
      CN=2.0*BN
 1030 IF(M)1031,1032,1032
 1032 P=P*PPP/((CN-3.0)*(CN-4.0)*(CN-5.0)*(CN-6.0))
      TERM=P*(1.0/(CN-3.0)-PP/(((CN-1.0)**2)*(CN-2.0)))
      IF(ABSF(TERM)-ERROR)1033,1033,1034
 1034 SI=SI+TERM
      GOTO1031
 1033 M=-1
 1031 IF(MM)1035,1036,1036
 1036 Q=Q*PPP/((CN-2.0)*(CN-3.0)*(CN-4.0)*(CN-5.0))
      TERM=-Q*(1.0/(CN-2.0)-PP/((CN-1.0)*CN**2))
      IF(ABSF(TERM)-ERROR)1037,1037,1038
 1038 CI=CI+TERM
      GOTO1035
 1037 MM=-1
 1035 IF(M)1040,1029,1029
 1029 CONTINUE
 1040 SSI=SI-1.57079633
      SNF=SINF(ED)
      CSF=COSF(ED)
      CF(1)=ED*(SNF*CI-CSF*SSI)
      CF(2)=-ED2*(CSF*CI+SNF*SSI)
      CF(3)=0.5*ED2*(1.0-CF(1))
      CF(4)=0.5*(CF(1)+CF(2))
      CF(5)=0.25*ED2*(CF(1)-CF(2))
      CF(6)=ED2*(CF(2)-CF(3))/6.0
      GOTO1050
C     SUBROUTINE FOR ARG BETWEEN 1 AND 18
 1025 A0(1)=38.102495
      A2(1)=335.67732
      A4(1)=265.187033
      A6(1)=38.027264
      A0(2)=157.105423
      A2(2)=570.236280
      A4(2)=322.624911
      A6(2)=40.021433
      A0(3)=21.821899
      A2(3)=352.018498
      A4(3)=302.757865
      A6(3)=42.242855
      A0(4)=449.690326
      A2(4)=1114.978885
      A4(4)=482.485984
      A6(4)=48.196927
 1045 REC=1.0/ED2
 1046 DO1047I=1,4
 1047 QP(I)=REC*(REC*(REC*A0(I)+A2(I))+A4(I))+A6(I)+ED2
 1048 CF(1)=QP(1)/QP(2)
      CF(2)=QP(3)/QP(4)
      CF(3)=0.5*ED2*(1.0-CF(1))
      CF(4)=0.5*(CF(1)+CF(2))
      CF(5)=0.25*ED2*(CF(1)-CF(2))
      CF(6)=ED2*(CF(2)-CF(3))/6.0
      GOTO1050
C     OUTPUT OF CF(J)
 1050 WRITEOUTPUTTAPE6,1082,ETA(IE),(CF(J),J=1,6)
C     CALCULATION OF W FOR ALPHA EQUALS 1.0,1.5
```

```
      W1(IE)=CST8*CR(9,IR)*(-2.*CF(2)+2.*CR(1,IR)*CF(1)
     1+(6.*CF(6)+2.*CR(3,IR)*CF(5))/(D*D)+CR(5,IR)*CF(3)/D
     2+CR(7,IR)*CF(4))/D
      W15(IE)=CST8*CR(9,IR)*(-3.*CF(2)+3.*CR(2,IR)*CF(1)
     1+(6.*CF(6)+2.*CR(4,IR)*CF(5))/(D*D)+CR(6,IR)*CF(3)/D
     2+CR(8,IR)*CF(4))/D
      WR1(IE,IR)=W1(IE)
      WR15(IE,IR)=W15(IE)
C     RCUTING AND OUTPUT OF W1, W15
 1051 IE=IE+1
      IF(17-IE)1052,1053,1053
 1052 WRITEOUTPUTTAPE6,1481,RHO(IR),(ETA(IE),WR1(IE,IR),
     1WR15(IE,IR),IE=1,17)
      IR=IR+1
      IF(NR-IR)1054,1055,1055
C     W1, W15 FOR RHO EQUALS 1.0
 1071 DO1073IE=1,17
      E=ETA(IE)
      W1(IE)=CST8*(PI*(CR(1,IR)+0.25*CR(3,IR)*E*E
     1+C.25*CR(7,IR))+0.5*E*CR(5,IR))*E
      W15(IE)=CST8*(PI*(1.5*CR(2,IR)+0.25*CR(4,IR)*E*E
     1+0.25*CR(8,IR))+0.5*E*CR(6,IR))*E
      WR1(IE,IR)=W1(IE)
      WR15(IE,IR)=W15(IE)
 1073 WRITEOUTPUTTAPE6,1482,E,WR1(IE,IR),WR15(IE,IR)
      IR=IR+1
      IF(NR-IR)1054,1055,1055
 1054 CONTINUE
 1081 FORMAT(1H1F8.3)
 1082 FORMAT(F8.3,6E17.8)
C     INTEGRATION OVER G(ALPHA, ETA), ETA(20, 100)
C     G(XI) FOR ALPHA EQUALS 1.0
      GL(1,1)=.39822131E-5
      GL(1,2)=.16191809E-5
      GL(1,3)=.77825388E-6
      GL(1,4)=.41943768E-6
      GL(1,5)=.24570713E-6
      GL(1,6)=.15336414E-6
      GL(1,7)=.10062709E-6
      GL(1,8)=.68741561E-7
      GL(1,9)=.48547497E-7
      GL(1,10)=.35255710E-7
      GL(1,11)=.26218130E-7
      GL(1,12)=.19900146E-7
      GL(1,13)=.15375868E-7
      GL(1,14)=.12067421E-7
      GL(1,15)=.96028132E-8
      GL(1,16)=.77364523E-8
      GL(1,17)=.63023160E-8
C     G(XI) FOR ALPHA EQUALS 1.5
      GL(2,1)=.10336716E-5
      GL(2,2)=.41716614E-6
      GL(2,3)=.19976667E-6
      GL(2,4)=.10745283E-6
      GL(2,5)=.62878519E-7
      GL(2,6)=.39223833E-7
      GL(2,7)=.25727630E-7
      GL(2,8)=.17572396E-7
      GL(2,9)=.12409252E-7
      GL(2,10)=.90115547E-8
      GL(2,11)=.67015941E-8
      GL(2,12)=.50868314E-8
```

```
      GL(2,13)=.39305232E-8
      GL(2,14)=.30849447E-8
      GL(2,15)=.24550185E-8
      GL(2,16)=.19779795E-8
      GL(2,17)=.16113982E-8
C     MULTIPLY BY WEIGHTING FACTORS
      DO1710J=1,2
C     INTERVAL (20, 100)
      WGL(J,1)=GL(J,1)/0.6
      DO1711IE=2,16,2
 1711 WGL(J,IE)=GL(J,IE)/0.15
      DO1712IE=3,15,2
 1712 WGL(J,IE)=GL(J,IE)/0.3
      WGL(J,17)=GL(J,17)/0.6
 1710 CONTINUE
      WRITEOUTPUTTAPE6,1483
      DO1404IR=1,NR
      TZZ1=0.0
      TZZ15=0.0
      UZ1=0.0
      UZ15=0.0
      DO1405IE=1,17
      TZZ1=TZZ1+WGL(1,IE)*E3J(IE,IR)*ETA(IE)
      TZZ15=TZZ15+WGL(2,IE)*E3J(IE,IR)*ETA(IE)
      UZ1=UZ1-WGL(1,IE)*E3J(IE,IR)
 1405 UZ15=UZ15-1.5*WGL(2,IE)*E3J(IE,IR)
      WRITEOUTPUTTAPE6,1484,RHO(IR),TZZ1,TZZ15,UZ1,UZ15
 1404 CONTINUE
      WRITEOUTPUTTAPE6,1485
      DO1406IR=1,NR
      TZZ1=0.0
      TZZ15=0.0
      DO1407IE=1,17
      TZZ1=TZZ1+WGL(1,IE)*WR1(IE,IR)
 1407 TZZ15=TZZ15+WGL(2,IE)*WR15(IE,IR)
      WRITEOUTPUTTAPE6,1486,RHO(IR),TZZ1,TZZ15
 1406 CONTINUE
 1480 FORMAT(F12.3,3E20.8)
 1481 FORMAT(1H1F8.3/(F12.3,2E20.8))
 1482 FORMAT(F12.3,2E20.8)
 1483 FORMAT(12H1PROGRAM 14./4H RHO/50H CONTRIBUTION OF J2 TERM TO TZZ1,
     1 TZZ15, UZ1, UZ15/13H ETA(20, 100))
 1484 FORMAT(F12.3/4E20.8)
 1485 FORMAT(12H1PROGRAM 14./4H RHO/51H CONTRIBUTION TO TZZ STRESS FROM
     1W(ETA, RHO, ALPHA)/13H ETA(20, 100))
 1486 FORMAT(F12.3/2E20.8)
      CALLSYSTEM
      END  (   1 ,   1 ,   0 ,   1 ,   0   )                   ANL-30
```

```
C       PROGRAM 11. BOUNDARY CONDITION AND UZ.
C       CALCULATION OF TERMS INVOLVING J2(ETA, RHO)
C       FOR ETA INTERVALS OF 0.01.
C       PARABOLIC INTERPOLATION USED FOR VALUES OF G(ETA).
        DIMENSIONGZ(2),G(2,40),RHO(10),X(2,2000),A(2,40),B(2,40),
       1C(2,40),TZZ(2,10),UZ(2,10),CRT(10),RF(7,10),TABJ(120),
       2TABY(120),TABS(120),TABX(120)
        READINPUTTAPE7,680,NR,(RHO(I),I=1,NR)
  680   FORMAT(I2/(F12.3))
        DO1150IR=1,NR
        R=RHO(IR)
        R2=R*R
        R3=R*R2
        R4=R2*R2
        CRT(IR)=.63661977/SQRTF(R)
        F12=((R-1.)/R2)*(-3.6328125*R+.8203125)
        F14=((R-1.)/R4)*(-.75942993*R3-4.0278625*R2
       X-1.0478210*R-.31723022)
        F16=((R-1.)/(R2*R4))*(1.7075908*R2*R3+3.41766*R4
       X+4.0406299*R3+3.5044026*R2+2.3519647*R+1.1276543)
        F21=(2.375*R-1.875)/R
        F23=(-1.7431641*R3+6.8115234*R2-1.9482422*R
       X-.30761719)/R3
        F25=(R4*(.91203690*R+1.4239311)+1.4299393*R3
       X+1.1175156*R2+.75342178*R+.51549912)/(R2*R3)
        F27=(R3*(-4.2488404*R4-3.2017329*R3-.74815527*R2
       X+.23361370*R-.55298433)-1.8727116*R2
       X-2.6781790*R-3.0809127)/(R3*R4)
        RF(1,IR)=F21
        RF(2,IR)=F12
        RF(3,IR)=F23
        RF(4,IR)=F14
        RF(5,IR)=F25
        RF(6,IR)=F16
        RF(7,IR)=F27
        TZZ(1,IR)=0.0
        TZZ(2,IR)=0.0
        UZ(1,IR)=0.0
        UZ(2,IR)=0.0
 1150   CONTINUE
C       G(1,ETA),G(1.5,ETA)FROM PROGRAM 8
C       G(XI) FOR ALPHA EQUALS 1.0
        G(1,1)=.20154125
        G(1,2)=.20993569
        G(1,3)=.21978752
        G(1,4)=.22626457
        G(1,5)=.21693650
        G(1,6)=.18155049
        G(1,7)=.13003021
        G(1,8)=.84137466E-1
        G(1,9)=.52481723E-1
        G(1,10)=.33041545E-1
        G(1,11)=.21327350E-1
        G(1,12)=.14224672E-1
        G(1,13)=.97755832E-2
        G(1,14)=.69195035E-2
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      G(1,15)=.50199737E-2
      G(1,16)=.37282833E-2
      G(1,17)=.28227726E-2
      G(1,18)=.21768700E-2
      G(1,19)=.17043766E-2
      G(1,20)=.13537167E-2
      G(1,21)=.88540018E-3
      G(1,22)=.60372003E-3
      G(1,23)=.42581421E-3
      G(1,24)=.30909523E-3
      G(1,25)=.22985333E-3
      G(1,26)=.17456127E-3
      G(1,27)=.13498336E-3
      G(1,28)=.10606523E-3
      G(1,29)=.84515418E-4
      G(1,30)=.68194988E-4
      G(1,31)=.45863959E-4
      G(1,32)=.31998922E-4
      G(1,33)=.23014008E-4
      G(1,34)=.16982160E-4
      G(1,35)=.12808580E-4
      G(1,36)=.98455451E-5
      G(1,37)=.76940523E-5
      G(1,38)=.61009600E-5
      G(1,39)=.49006505E-5
      G(1,40)=.39822131E-5
C     G(XI) FOR ALPHA EQUALS 1.5
      G(2,1)=.97895779E-1
      G(2,2)=.98713586E-1
      G(2,3)=.10012834
      G(2,4)=.99914737E-1
      G(2,5)=.92933235E-1
      G(2,6)=.75519875E-1
      G(2,7)=.52592502E-1
      G(2,8)=.33125947E-1
      G(2,9)=.20145591E-1
      G(2,10)=.12380441E-1
      G(2,11)=.78127421E-2
      G(2,12)=.51004198E-2
      G(2,13)=.34359130E-2
      G(2,14)=.23866579E-2
      G(2,15)=.17014134E-2
      G(2,16)=.12429380E-2
      G(2,17)=.92674737E-3
      G(2,18)=.70445783E-3
      G(2,19)=.54421783E-3
      G(2,20)=.42685848E-3
      G(2,21)=.27293737E-3
      G(2,22)=.18244641E-3
      G(2,23)=.12646843E-3
      G(2,24)=.90412962E-4
      G(2,25)=.66339907E-4
      G(2,26)=.49789584E-4
      G(2,27)=.38101513E-4
      G(2,28)=.29663067E-4
      G(2,29)=.23443080E-4
      G(2,30)=.18778225E-4
      G(2,31)=.12473916E-4
      G(2,32)=.86167022E-5
      G(2,33)=.61473403E-5
      G(2,34)=.45062138E-5
      G(2,35)=.33802395E-5
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      G(2,36)=.25865050E-5
      G(2,37)=.20136259E-5
      G(2,38)=.15915941E-5
      G(2,39)=.12750054E-5
      G(2,40)=.10336716E-5
C     CALCULATION OF INTERPOLATION CONSTANTS.
      GZ(1)=.19634954
      GZ(2)=.09817477
      DO1101J=1,2
C     INTERVAL (0,.5)
      FD=GZ(J)-2.*G(J,1)+G(J,2)
      A(J,1)=8.*FD
      B(J,1)=-4.*FD+2.*(G(J,2)-GZ(J))
      C(J,1)=0.5*(FD+GZ(J)-G(J,2))+G(J,1)
      DO1102IE=1,50
      E=0.01*FLOATF(IE)
 1102 X(J,IE)=(A(J,1)*E*E+B(J,1)*E+C(J,1))*(E**4)
C     INTERVALS (.5,1), (1,1.5),(1.5,2)
      DO1103K=3,7,2
      FD=G(J,K-1)-2.*G(J,K)+G(J,K+1)
      A(J,K)=8.*FD
      X2=0.25*FLOATF(K)
      B(J,K)=-16.*X2*FD+2.*(G(J,K+1)-G(J,K-1))
      C(J,K)=8.*X2*X2*FD+2.*X2*(G(J,K-1)-G(J,K+1))+G(J,K)
      IK=25*K-24
      IKK=25*K+25
      DO1104IE=IK,IKK
      E=0.01*FLOATF(IE)
 1104 X(J,IE)=(A(J,K)*E*E+B(J,K)*E+C(J,K))*(E**4)
 1103 CONTINUE
C     INTERVALS (2, 2.5) THROUGH (4.5, 5)
      DO1105K=9,19,2
      X2=0.25*FLOATF(K)
      F2=G(J,K)*(X2**4)
      F1=G(J,K-1)*((X2-0.25)**4)
      F3=G(J,K+1)*((X2+0.25)**4)
      FD=F1-2.*F2+F3
      A(J,K)=8.*FD
      B(J,K)=-16.*X2*FD+2.*(F3-F1)
      C(J,K)=8.*X2*X2*FD+2.*X2*(F1-F3)+F2
      IK=25*K-24
      IKK=25*K+25
      DO1106IE=IK,IKK
      E=0.01*FLOATF(IE)
 1106 X(J,IE)=E*(A(J,K)*E+B(J,K))+C(J,K)
 1105 CONTINUE
C     INTERVALS (5, 6) THROUGH (9, 10)
      DO 1107K=21,29,2
      X2=0.5*FLOATF(K-10)
      F2=G(J,K)*(X2**4)
      F1=G(J,K-1)*((X2-0.5)**4)
      F3=G(J,K+1)*((X2+0.5)**4)
      FD=F1-2.*F2+F3
      A(J,K)=2.*FD
      B(J,K)=-4.*X2*FD+F3-F1
      C(J,K)=2.*X2*X2*FD+X2*(F1-F3)+F2
      IK=50*K-549
      IKK=50*K-450
      DO1108IE=IK,IKK
      E=0.01*FLOATF(IE)
 1108 X(J,IE)=E*(A(J,K)*E+B(J,K))+C(J,K)
 1107 CONTINUE
```

```
C     INTERVALS (10, 12) THROUGH (18, 20)
      DO1109K=31,39,2
      X2=K-20
      F2=G(J,K)*(X2**4)
      F1=G(J,K-1)*((X2-1.0)**4)
      F3=G(J,K+1)*((X2+1.0)**4)
      FD=F1-2.*F2+F3
      A(J,K)=0.5*FD
      B(J,K)=-X2*FD+0.5*(F3-F1)
      C(J,K)=0.5*X2*X2*FD+0.5*X2*(F1-F3)+F2
      IK=100*K-2099
      IKK=100*K-1900
      DO1110IE=IK,IKK
      E=0.01*FLOATF(IE)
 1110 X(J,IE)=E*(A(J,K)*E+B(J,K))+C(J,K)
 1109 CONTINUE
 1101 CONTINUE
      WRITEOUTPUTTAPE6,1180
 1180 FORMAT(70H1K, INTERPOLATION CONSTANTS FOR INTERVAL CENTERED ON G(J
     1, K), J = 1, 2)
      DO1111K=1,39,2
      WRITEOUTPUTTAPE6,1181,K,A(1,K),B(1,K),C(1,K),A(2,K),B(2,K),C(2,K)
 1181 FORMAT(I2/3E20.8/3E20.8)
 1111 CONTINUE
      DO1112IE=1,2000
      E=0.01*FLOATF(IE)
      Z1=BESJF(E,0.,1,120,XLOCF(TABJ))
      Z2=BESYF(E,0.,1,120,XLOCF(TABY))
      AJ0=TABJ(1)
      AJ1=TABJ(2)
      AY0=TABY(1)
      AY1=TABY(2)
      AJ2PR=(1.0-4.0/(E*E))*AJ1+2.*AJ0/E
      AY2PR=(1.0-4.0/(E*E))*AY1+2.*AY0/E
      DO1113IR=1,NR
      R=RHO(IR)
      RE=E*R
      IF(20.-RE)604,605,605
  605 Z3=BESJF(RE,0.,1,120,XLOCF(TABS))
      Z4=BESYF(RE,0.,1,120,XLOCF(TABX))
      AJR0=TABS(1)
      AJR1=TABS(2)
      AYR0=TABX(1)
      AYR1=TABX(2)
      AJ2=2.*AJR1/RE-AJR0
      AY2=2.*AYR1/RE-AYR0
      AJ=AY2PR*AJ2-AJ2PR*AY2
      GOTO606
  604 SN=SINF(E*(R-1.))
      CS=COSF(E*(R-1.))
      E2=E*E
      E4=E2*E2
      AJ=CRT(IR)*(CS*(1.+(RF(2,IR)+RF(4,IR)/E2
     1+RF(6,IR)/E4)/E2)/E+(SN/E2)*(RF(1,IR)
     2+(RF(3,IR)+RF(5,IR)/E2+RF(7,IR)/E4)/E2))
  606 IF(1999-IE)1114,1115,1115
C     INTEGRATION USING TRAPEZOIDAL RULE
 1115 TZZ(1,IR)=TZZ(1,IR)+AJ*X(1,IE)
      TZZ(2,IR)=TZZ(2,IR)+AJ*X(2,IE)
      UZ(1,IR)=UZ(1,IR)+AJ*X(1,IE)/E
      UZ(2,IR)=UZ(2,IR)+AJ*X(2,IE)/E
      GOTO1113
```

```
 1114 TZZ(1,IR)=(TZZ(1,IR)+AJ*X(1,IE)*0.5)*0.01
      TZZ(2,IR)=(TZZ(2,IR)+AJ*X(2,IE)*0.5)*0.01
      UZ(1,IR)=(UZ(1,IR)+AJ*X(1,IE)*0.5/E)*0.01
      UZ(2,IR)=(UZ(2,IR)+AJ*X(2,IE)*0.5/E)*0.01
 1113 CONTINUE
 1112 CONTINUE
      WRITEOUTPUTTAPE6,1182
 1182 FORMAT(4H1RHO/64H CONTRIBUTION OF J2 TERM TO TZZ1, TZZ15, UZ1, UZ1
     15 FOR LARGE RHO)
      DO1116IR=1,NR
      WRITEOUTPUTTAPE6,1183,RHO(IR),TZZ(1,IR),
     1TZZ(2,IR),UZ(1,IR),UZ(2,IR)
 1183 FORMAT(F12.3/4E20.8)
 1116 CONTINUE
      CALL SYSTEM
      END  (   1 ,   1 ,   0 ,   1 ,   0   )                        ANL-30
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```
C     PROGRAM 7. STRESSES ON HOLE IN HALF-SPACE
C     FOR VARIOUS ZETAS, ALPHA IS 1, 1.5.
C     SUM OF PROGAMS 2,3,4, MULTIPLIED BY G AND INTEGRATED.
      DIMENSIONETA(40),S(40,15,6),T(6),ZE(15),V(6),
     1TABR(66),G(2,40),STRS(6)
C     INPUT OF ZETAS
      READINPUTTAPE7,380,NZ,(ZE(I),I=1,NZ)
  380 FORMAT(I2/(F12.3))
C     STORING OF ETAS
      DO701IE=1,20
  701 ETA(IE)=0.25*FLOATF(IE)
      DO702IE=21,30
  702 ETA(IE)=-5.+0.5*FLOATF(IE)
      DO703IE=31,40
  703 ETA(IE)=IE-20
      CST=2.7018982E-2
      CST2=5.4037965E-2
C     INTEGRATION OVER GAMMA FROM 0 TO 10
C     INITIALIZATION OF S
      DO70IZ=1,NZ
      DO704IE=1,40
      S(IE,IZ,1)=-0.5
      S(IE,IZ,2)=-2.5
      S(IE,IZ,3)=0.75
      S(IE,IZ,4)=0.625
      S(IE,IZ,5)=6.*ZE(IZ)
  704 S(IE,IZ,6)=12.*ZE(IZ)
   70 CONTINUE
C     ROUTING
      JG=1
      IG=1
      GOTO3
  134 IG=IG+2
      IF(100-IG)140,136,3
  140 JG=2
      IG=2
      GOTO3
  136 JG=3
      GOTO3
   10 IE=IE+1
      IF(40-IE)131,705,705
  131 IZ=IZ+1
      IF(NZ-IZ)133,132,132
  133 GOTO(134,134,13),JG
C     FUNCTIONS OF GAMMA
    3 G=0.1*FLOATF(IG)
      G2=G*G
      Z1=BESKF(G,0.,1,66,XLOCF(TABR))
      AK0=TABR(1)
      AK1=TABR(2)
      AK=-(2.+(G2*AK1)/(G*AK0+2.*AK1))
      AKSQ=AK*AK
      FFG=G2*(G2+8.-2.*AK)+12.
      FFG1=G2+4.-AKSQ
      HG=G2-(G2+3.)*AK
      DELTA2=-4.*G2+AK*(8.*G2+24.-G2*AK-6.*AKSQ)
      DEL1=FFG1*FFG+DELTA2
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      DEL15=FFG1*FFG+1.5*DELTA2
      TZ11=-1.+(G2/DEL1)*(-G2*(G2+9.)-12.+(2.*G2+3.)*AK
     X+(G2+3.)*AKSQ)
      TZ21=G2+2.5+(G2/DEL1)*((FFG1-2.*AK)*FFG-4.*HG)
      TZ115=-1.+(G2/DEL15)*(-G2*(G2+9.)-12.+(2.*G2+1.5)*AK
     X+(G2+3.)*AKSQ)
      TZ215=G2-0.5*G+0.75+(G2/DEL15)*((FFG1-2.*AK)*FFG
     X-6.*HG)
      TD11=-3.*(1.+G2*HG/DEL1)
      TD21=3.*(G2-0.5*G+3.25-G2*AK*FFG/DEL1)
      TD115=-2.5*(1.+G2*HG/DEL15)
      TD215=2.5*(G2-G+3.75-G2*AK*FFG/DEL15)
      TT11=-1.+1.5/G+(G/DEL1)*(4.*G2*(G2+7.)+48.+AK*
     X(-G2*(G2+4.)+12.)-AKSQ*(4.*G2+12.)+(G2-3.)*AK*AKSQ)
      TT21=-0.5+2.25/G+(G/DEL1)*(-FFG1*FFG+(4.-AKSQ)*HG)
      TT115=-1.+3.5/G+(G/DEL15)*(4.*G2*(G2+7.)+48.+AK*
     X(-G2*(G2+4.)+18.)-AKSQ*(4.*G2+12.)+(G2-4.5)*AK*AKSQ)
      TT215=-0.5*G+3.375/G+(G/DEL15)*(-FFG1*FFG+(6.-1.5*AKSQ)*HG)
      IZ=1
  132 SN=SINF(G*ZE(IZ))
      CS=COSF(G*ZE(IZ))
      IE=1
  705 E=ETA(IE)*ETA(IE)/(ETA(IE)*ETA(IE)+G2)
      T(1)=E*(2.*TZ11+E*TZ21)*CS
      T(2)=E*(3.*TZ115+E*TZ215)*CS
      T(3)=E*(2.*TD11+E*TD21)*CS
      T(4)=E*(3.*TD115+E*TD215)*CS
      T(5)=E*(TT11+2.*E*TT21)*SN
      T(6)=E*(1.5*TT115+2.*E*TT215)*SN
      GOTO(50,51,52),JG
C     SUMMATIONS
   50 DO60IS=1,6
   60 S(IE,IZ,IS)=S(IE,IZ,IS)+4.*T(IS)
      GOTO10
   51 DO61IS=1,6
   61 S(IE,IZ,IS)=S(IE,IZ,IS)+2.*T(IS)
      GOTO10
   52 DO62IS=1,6
   62 S(IE,IZ,IS)=CST*(S(IE,IZ,IS)+T(IS))
      GOTO10
C     INTEGRATION OVER GAMMA FROM 10 TO 100
C     GAMMA ROUTING
   13 JG=1
      G=10.
      GOTO303
  311 JG=2
      IG=1
      GOTO312
  315 IG=IG+2
      IF(450-IG)316,319,312
  316 JG=3
      IG=2
      GOTO312
  319 JG=4
      GOTO312
C     ETA LOOPING
  306 IE=IE+1
      IF(40-IE)308,307,307
C     ZETA LOOPING
  308 IZ=IZ+1
      IF(NZ-IZ)310,309,309
  310 GOTO(311,315,315,322),JG
```

```
C     FUNCTIONS OF GAMMA IN INTEGRANDS
C     ASYMPTOTIC EXPANSIONS
  312 G=10.+0.2*FLOATF(IG)
  303 G2=G*G
      G3=G2*G
      D1=-G2*G3-3.*G2*G2-10.125*G3-6.75*G2
     X+5.4140625*G-16.875+8.7451172/G+84.375/G2
      D15=-G2*G3-3.5*G2*G2-11.625*G3-8.25*G2
     X+5.9765625*G-16.875+12.084961/G+74.53125/G2
      TZ11=(4.5*G3+12.75*G2-21.9375*G+25.3125
     X+29.355469/G-274.21875/G2)/D1
      TZ21=(-3.75*G2*G2+13.5*G3-49.3125*G2
     X+32.34375*G+160.3125)/D1
      TZ115=(0.5*G2*G2+7.5*G3+15.*G2-19.6875*G
     X+22.5+27.597656/G-262.26563/G2)/D15
      TZ215=(2.4375*G2*G2+25.03125*G3-40.863281*G2
     X+21.708984*G+182.39502)/D15
      TD11=(4.5*G2*G2+15.75*G3+21.375*G2-36.28125*G
     X+63.28125-13.974609/G-329.0625/G2)/D1
      TD21=(-3.9375*G2*G2+0.5625*G3-124.83984*G2
     X+156.41016*G+71.257324)/D1
      TD115=(5.*G2*G2+16.875*G3+21.5625*G2-31.640625*G
     X+52.734375-19.995117/G-249.60938/G2)/D15
      TD215=(0.9375*G2*G2-12.65625*G3-134.70703*G2
     X+171.82617*G-5.6030273)/D15
      TT11=(10.125*G3-6.75*G2-32.414063*G+61.382813
     X-74.838867/G-95.185547/G2)/D1
      TT21=(-0.5625*G3-30.9375*G2+3.1992188*G
     X+18.246094-166.09131/G)/D1
      TT115=(6.125*G3-27.1875*G2-54.539063*G
     X+77.167969-115.44434/G-59.919434/G2)/D15
      TT215=(-7.875*G3-57.375*G2+3.796875*G
     X+11.601563-227.8125/G)/D15
C     CALCULATION OF COMPLETE INTEGRAND INVOLVING ZETA, ETA
      IZ=1
  309 SN=SINF(G*ZE(IZ))
      CS=COSF(G*ZE(IZ))
      IE=1
  307 E=ETA(IE)*ETA(IE)/(ETA(IE)*ETA(IE)+G2)
      T(1)=E*(2.*TZ11+E*TZ21)*CS
      T(2)=E*(3.*TZ115+E*TZ215)*CS
      T(3)=E*(2.*TD11+E*TD21)*CS
      T(4)=E*(3.*TD115+E*TD215)*CS
      T(5)=E*(TT11+2.*E*TT21)*SN
      T(6)=E*(1.5*TT115+2.*E*TT215)*SN
      GOTO(706,707,708,709),JG
C     SUMMATIONS
  706 DO710IS=1,6
  710 S(IE,IZ,IS)=S(IE,IZ,IS)+CST2*T(IS)
      GOTO306
  707 DO711IS=1,6
  711 S(IE,IZ,IS)=S(IE,IZ,IS)+4.*CST2*T(IS)
      GOTO306
  708 DO712IS=1,6
  712 S(IE,IZ,IS)=S(IE,IZ,IS)+2.*CST2*T(IS)
      GOTO306
  709 DO713IS=1,6
  713 S(IE,IZ,IS)=S(IE,IZ,IS)+CST2*T(IS)
      GOTO306
  322 DO714IZ=1,NZ
      WRITEOUTPUTTAPE6,780,ZE(IZ),(ETA(IE),(S(IE,IZ,IS),
     XIS=1,6),IE=1,40)
```

```
  714 CONTINUE
  780 FORMAT(1H1F7.3/(F8.3,6E17.8))
C       TERMS IN ETA
        C1=.63661977
        DO403IZ=1,NZ
        ZT=ZE(IZ)
        WRITEOUTPUTTAPE6,481,ZT
  481 FORMAT(1H1F7.3)
        DO715IE=1,40
        ET=ETA(IE)
        EEZ=C1*EXPF(-ZT*ET)
        V(1)=EEZ*(2.*ZT*ET**4-2.5*ET*ET*ZT+1.5*ET)
        V(2)=EEZ*(2.*ZT*ET**4-0.75*ET*ET*ZT+5.25*ET)
        V(3)=EEZ*(3.*ZT*ET**4-9.75*ET*ET*ZT+2.25*ET)
        V(4)=EEZ*(2.5*ZT*ET**4-9.375*ET*ET*ZT+5.625*ET)
        V(5)=EEZ*(2.*ZT*ET**3+4.5*ET*ZT)-12.*(C1-EEZ)
        V(6)=EEZ*(3.*ZT*ET**3+6.75*ET*ZT)-24.*(C1-EEZ)
        DO716IS=1,6
  716 S(IE,IZ,IS)=V(IS)+S(IE,IZ,IS)
        WRITEOUTPUTTAPE6,781,ET,(S(IE,IZ,IS),IS=1,6)
  781 FORMAT(F8.3,6E17.8)
  715 CONTINUE
  403 CONTINUE
        WRITEOUTPUTTAPE6,790
  790 FORMAT(1H1)
C       INTEGRATION OVER G(ALPHA,ETA)
C       G(1,ETA),G(1.5,ETA)FROM PROGRAM 8
C       G(XI) FOR ALPHA EQUALS 1.0
        G(1,1)=.20154125
        G(1,2)=.20993569
        G(1,3)=.21978752
        G(1,4)=.22626457
        G(1,5)=.21693650
        G(1,6)=.18155049
        G(1,7)=.13003021
        G(1,8)=.84137466E-1
        G(1,9)=.52481723E-1
        G(1,10)=.33041545E-1
        G(1,11)=.21327350E-1
        G(1,12)=.14224672E-1
        G(1,13)=.97755832E-2
        G(1,14)=.69195035E-2
        G(1,15)=.50199737E-2
        G(1,16)=.37282833E-2
        G(1,17)=.28227726E-2
        G(1,18)=.21768700E-2
        G(1,19)=.17043766E-2
        G(1,20)=.13537167E-2
        G(1,21)=.88540018E-3
        G(1,22)=.60372003E-3
        G(1,23)=.42581421E-3
        G(1,24)=.30909523E-3
        G(1,25)=.22985333E-3
        G(1,26)=.17456127E-3
        G(1,27)=.13498336E-3
        G(1,28)=.10606523E-3
        G(1,29)=.84515418E-4
        G(1,30)=.68194988E-4
        G(1,31)=.45863959E-4
        G(1,32)=.31998922E-4
        G(1,33)=.23014008E-4
        G(1,34)=.16982160E-4
```

```
      G(1,35)=.12808580E-4
      G(1,36)=.98455451E-5
      G(1,37)=.76940523E-5
      G(1,38)=.61009600E-5
      G(1,39)=.49006505E-5
      G(1,40)=.39822131E-5
C     G(XI) FOR ALPHA EQUALS 1.5
      G(2,1)=.97895779E-1
      G(2,2)=.98713586E-1
      G(2,3)=.10012834
      G(2,4)=.99914737E-1
      G(2,5)=.92933235E-1
      G(2,6)=.75519875E-1
      G(2,7)=.52592502E-1
      G(2,8)=.33125947E-1
      G(2,9)=.20145591E-1
      G(2,10)=.12380441E-1
      G(2,11)=.78127421E-2
      G(2,12)=.51004198E-2
      G(2,13)=.34359130E-2
      G(2,14)=.23866579E-2
      G(2,15)=.17014134E-2
      G(2,16)=.12429380E-2
      G(2,17)=.92674737E-3
      G(2,18)=.70445783E-3
      G(2,19)=.54421783E-3
      G(2,20)=.42685848E-3
      G(2,21)=.27293737E-3
      G(2,22)=.18244641E-3
      G(2,23)=.12646843E-3
      G(2,24)=.90412962E-4
      G(2,25)=.66339907E-4
      G(2,26)=.49789584E-4
      G(2,27)=.38101513E-4
      G(2,28)=.29663067E-4
      G(2,29)=.23443080E-4
      G(2,30)=.18778225E-4
      G(2,31)=.12473916E-4
      G(2,32)=.86167022E-5
      G(2,33)=.61473403E-5
      G(2,34)=.45062138E-5
      G(2,35)=.33802395E-5
      G(2,36)=.25865050E-5
      G(2,37)=.20136259E-5
      G(2,38)=.15915941E-5
      G(2,39)=.12750054E-5
      G(2,40)=.10336716E-5
C     MULTIPLY BY WEIGHTING FACTORS
      DO730J=1,2
      DO720IE=1,19,2
  720 G(J,IE)=G(J,IE)/3.
      DO721IE=2,18,2
  721 G(J,IE)=G(J,IE)/6.
      G(J,20)=G(J,20)/4.
      DO722IE=21,29,2
  722 G(J,IE)=G(J,IE)/1.5
      DO723IE=22,28,2
  723 G(J,IE)=G(J,IE)/3.
      G(J,30)=0.5*G(J,30)
      DO724IE=31,39,2
  724 G(J,IE)=G(J,IE)/.75
      DO725IE=32,38,2
```

```
  725 G(J,IE)=G(J,IE)/1.5
  730 G(J,40)=G(J,40)/3.
      WRITEOUTPUTTAPE6,782,(ETA(IE),G(1,IE),G(2,IE),IE=1,40)
  782 FORMAT(F7.3,2E20.8)
      WRITEOUTPUTTAPE6,790
      DO726IZ=1,NZ
      DO727IS=1,6
  727 STRS(IS)=0.0
      DO728IE=1,40
      STRS(1)=STRS(1)+G(1,IE)*S(IE,IZ,1)
      STRS(2)=STRS(2)+G(2,IE)*S(IE,IZ,2)
      STRS(3)=STRS(3)+G(1,IE)*S(IE,IZ,3)
      STRS(4)=STRS(4)+G(2,IE)*S(IE,IZ,4)
      STRS(5)=STRS(5)+G(1,IE)*S(IE,IZ,5)
  728 STRS(6)=STRS(6)+G(2,IE)*S(IE,IZ,6)
  726 WRITEOUTPUTTAPE6,783,ZT,(STRS(IS),IS=1,6)
  783 FORMAT(F7.3/6E17.8)
      CALLSYSTEM
      END  (   1 ,   1 ,   0 ,   1 ,   0   )                          ANL-30
```

```
C       PROGRAM 3A. EVALUATION OF STRESS INTEGRALS
C       FOR GAMMA (100,2000) IN STEPS OF 5.0, FOR SMALL ZETAS.
        DIMENSIONS(40,5,6),T(6),ETA(40),ZE(5),G(2,40),STRS(6)
C       INPUT OF ZETAS
        READINPUTTAPE7,380,NZ,(ZE(I),I=1,NZ)
  380   FORMAT(I2/(F12.3))
C       STORING OF ETAS
        DO701IE=1,20
  701   ETA(IE)=0.25*FLOATF(IE)
        DO702IE=21,30
  702   ETA(IE)=-5.+0.5*FLOATF(IE)
        DO703IE=31,40
  703   ETA(IE)=IE-20
        CST2=1.35094912
C       GAMMA ROUTING
        JG=1
        G=100.
        GOTO303
  311   JG=2
        IG=105
        GOTO312
  315   IG=IG+10
        IF(2000-IG)316,319,312
  316   JG=3
        IG=110
        GOTO312
  319   JG=4
        GOTO312
C       ETA LOOPING
  306   IE=IE+1
        IF(40-IE)308,307,307
C       ZETA LOOPING
  308   IZ=IZ+1
        IF(NZ-IZ)310,309,309
  310   GOTO(311,315,315,322),JG
C       FUNCTIONS OF GAMMA IN INTEGRANDS
C       ASYMPTOTIC EXPANSIONS
  312   G=IG
  303   G2=G*G
        G3=G2*G
        D1=-G2*G3-3.*G2*G2-10.125*G3-6.75*G2
       X+5.4140625*G-16.875+8.7451172/G+84.375/G2
        D15=-G2*G3-3.5*G2*G2-11.625*G3-8.25*G2
       X+5.9765625*G-16.875+12.084961/G+74.53125/G2
        TZ11=(4.5*G3+12.75*G2-21.9375*G+25.3125
       X+29.355469/G-274.21875/G2)/D1
        TZ21=(-3.75*G2*G2+13.5*G3-49.3125*G2
       X+32.34375*G+160.3125)/D1
        TZ115=(0.5*G2*G2+7.5*G3+15.*G2-19.6875*G
       X+22.5+27.597656/G-262.26563/G2)/D15
        TZ215=(2.4375*G2*G2+25.03125*G3-40.863281*G2
       X+21.708984*G+182.39502)/D15
        TC11=(4.5*G2*G2+15.75*G3+21.375*G2-36.28125*G
       X+63.28125-13.974609/G-329.0625/G2)/D1
        TC21=(-3.9375*G2*G2+0.5625*G3-124.83984*G2
       X+156.41016*G+71.257324)/D1
        TC115=(5.*G2*G2+16.875*G3+21.5625*G2-31.640625*G
```

```
      X+52.734375-19.995117/G-249.60938/G2)/D15
       TD215=(0.9375*G2*G2-12.65625*G3-134.70703*G2
      X+171.82617*G-5.6030273)/D15
       TT11=(10.125*G3-6.75*G2-32.414063*G+61.382813
      X-74.838867/G-95.185547/G2)/D1
       TT21=(-0.5625*G3-30.9375*G2+3.1992188*G
      X+18.246094-166.09131/G)/D1
       TT115=(6.125*G3-27.1875*G2-54.539063*G
      X+77.167969-115.44434/G-59.919434/G2)/D15
       TT215=(-7.875*G3-57.375*G2+3.796875*G
      X+11.601563-227.8125/G)/D15
C      CALCULATION OF COMPLETE INTEGRAND INVOLVING ZETA, ETA
       IZ=1
  309  SN=SINF(G*ZE(IZ))
       CS=COSF(G*ZE(IZ))
       IE=1
  307  E=ETA(IE)*ETA(IE)/(ETA(IE)*ETA(IE)+G2)
       T(1)=E*(2.*TZ11+E*TZ21)*CS
       T(2)=E*(3.*TZ115+E*TZ215)*CS
       T(3)=E*(2.*TD11+E*TD21)*CS
       T(4)=E*(3.*TD115+E*TD215)*CS
       T(5)=E*(TT11+2.*E*TT21)*SN
       T(6)=E*(1.5*TT115+2.*E*TT215)*SN
       GOTO(304,313,317,320),JG
C      SUMMATIONS
  304  DO305IS=1,6
  305  S(IE,IZ,IS)=T(IS)
       GOTO306
  313  DO314IS=1,6
  314  S(IE,IZ,IS)=S(IE,IZ,IS)+4.*T(IS)
       GOTO306
  317  DO318IS=1,6
  318  S(IE,IZ,IS)=S(IE,IZ,IS)+2.*T(IS)
       GOTO306
  320  DO321IS=1,6
  321  S(IE,IZ,IS)=CST2*(S(IE,IZ,IS)+T(IS))
       GOTO306
C      OUTPUT
  322  DO714IZ=1,NZ
       WRITEOUTPUTTAPE6,780,ZE(IZ),(ETA(IE),(S(IE,IZ,IS),
      XIS=1,6),IE=1,40)
  714  CONTINUE
  780  FORMAT(1H1F7.3/(F8.3,6E17.8))
       WRITEOUTPUTTAPE6,790
  790  FORMAT(1H1)
C      INTEGRATION OVER G(ALPHA,ETA)
C      G(1,ETA),G(1.5,ETA)FROM PROGRAM 8
C      G(XI) FOR ALPHA EQUALS 1.0
       G(1,1)=.20154125
       G(1,2)=.20993569
       G(1,3)=.21978752
       G(1,4)=.22626457
       G(1,5)=.21693650
       G(1,6)=.18155049
       G(1,7)=.13003021
       G(1,8)=.84137466E-1
       G(1,9)=.52481723E-1
       G(1,10)=.33041545E-1
       G(1,11)=.21327350E-1
       G(1,12)=.14224672E-1
       G(1,13)=.97755832E-2
       G(1,14)=.69195035E-2
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```
      G(1,15)=.50199737E-2
      G(1,16)=.37282833E-2
      G(1,17)=.28227726E-2
      G(1,18)=.21768700E-2
      G(1,19)=.17043766E-2
      G(1,20)=.13537167E-2
      G(1,21)=.88540018E-3
      G(1,22)=.60372003E-3
      G(1,23)=.42581421E-3
      G(1,24)=.30909523E-3
      G(1,25)=.22985333E-3
      G(1,26)=.17456127E-3
      G(1,27)=.13498336E-3
      G(1,28)=.10606523E-3
      G(1,29)=.84515418E-4
      G(1,30)=.68194988E-4
      G(1,31)=.45863959E-4
      G(1,32)=.31998922E-4
      G(1,33)=.23014008E-4
      G(1,34)=.16982160E-4
      G(1,35)=.12808580E-4
      G(1,36)=.98455451E-5
      G(1,37)=.76940523E-5
      G(1,38)=.61009600E-5
      G(1,39)=.49006505E-5
      G(1,40)=.39822131E-5
C     G(XI) FOR ALPHA EQUALS 1.5
      G(2,1)=.97895779E-1
      G(2,2)=.98713586E-1
      G(2,3)=.10012834
      G(2,4)=.99914737E-1
      G(2,5)=.92933235E-1
      G(2,6)=.75519875E-1
      G(2,7)=.52592502E-1
      G(2,8)=.33125947E-1
      G(2,9)=.20145591E-1
      G(2,10)=.12380441E-1
      G(2,11)=.78127421E-2
      G(2,12)=.51004198E-2
      G(2,13)=.34359130E-2
      G(2,14)=.23866579E-2
      G(2,15)=.17014134E-2
      G(2,16)=.12429380E-2
      G(2,17)=.92674737E-3
      G(2,18)=.70445783E-3
      G(2,19)=.54421783E-3
      G(2,20)=.42685848E-3
      G(2,21)=.27293737E-3
      G(2,22)=.18244641E-3
      G(2,23)=.12646843E-3
      G(2,24)=.90412962E-4
      G(2,25)=.66339907E-4
      G(2,26)=.49789584E-4
      G(2,27)=.38101513E-4
      G(2,28)=.29663067E-4
      G(2,29)=.23443080E-4
      G(2,30)=.18778225E-4
      G(2,31)=.12473916E-4
      G(2,32)=.86167022E-5
      G(2,33)=.61473403E-5
      G(2,34)=.45062138E-5
      G(2,35)=.33602395E-5
```

```
      G(2,36)=.25865050E-5
      G(2,37)=.20136259E-5
      G(2,38)=.15915941E-5
      G(2,39)=.12750054E-5
      G(2,40)=.10336716E-5
C     MULTIPLY BY WEIGHTING FACTORS
      DO730J=1,2
      DO720IE=1,19,2
  720 G(J,IE)=G(J,IE)/3.
      DO721IE=2,18,2
  721 G(J,IE)=G(J,IE)/6.
      G(J,20)=G(J,20)/4.
      DO722IE=21,29,2
  722 G(J,IE)=G(J,IE)/1.5
      DO723IE=22,28,2
  723 G(J,IE)=G(J,IE)/3.
      G(J,30)=0.5*G(J,30)
      DO724IE=31,39,2
  724 G(J,IE)=G(J,IE)/.75
      DO725IE=32,38,2
  725 G(J,IE)=G(J,IE)/1.5
  730 G(J,40)=G(J,40)/3.
      DO726IZ=1,NZ
      DO727IS=1,6
  727 STRS(IS)=0.0
      DO728IE=1,40
      STRS(1)=STRS(1)+G(1,IE)*S(IE,IZ,1)
      STRS(2)=STRS(2)+G(2,IE)*S(IE,IZ,2)
      STRS(3)=STRS(3)+G(1,IE)*S(IE,IZ,3)
      STRS(4)=STRS(4)+G(2,IE)*S(IE,IZ,4)
      STRS(5)=STRS(5)+G(1,IE)*S(IE,IZ,5)
  728 STRS(6)=STRS(6)+G(2,IE)*S(IE,IZ,6)
  726 WRITEOUTPUTTAPE6,783,ZE(IZ),(STRS(IS),IS=1,6)
  783 FORMAT(F7.3/6E17.8)
      CALLSYSTEM
      END  (  1,   1,   0,   1,   0  )                                  ANL-30
```

```
C      PROGRAM 17. STRESSES ON HOLE IN HALF-SPACE.
C      INTEGRALS OF G(ALPHA, ETA)XV(ETA, ZETA, ALPHA)
C      FOR LARGE ETA AND SMALL ZETA
       DIMENSIONV(36,10,6),ZE(10),ETA(36),GL(2,36),STRS(4,6),WGL(2,38)
       C1=.63661977
       READINPUTTAPE7,1780,NZ,(ZE(IZ),IZ=1,NZ)
 1780  FORMAT(I2/(F12.3))
C      STORING OF ETAS
       DO1701IE=1,17
 1701  ETA(IE)=15+5*IE
       DO1702IE=18,36
 1702  ETA(IE)=(IE-16)*100
       DO1703IZ=1,NZ
       Z=ZE(IZ)
       WRITEOUTPUTTAPE6,1781,Z
 1781  FORMAT(1H1F8.3)
       DO1704IE=1,36
       E=ETA(IE)
       IF(75.-E*Z)1703,1706,1706
 1706  EEZ=C1*EXPF(-E*Z)
       V(IE,IZ,1)=EEZ*(2.*Z*E**4-2.5*E*E*Z+1.5*E)
       V(IE,IZ,2)=EEZ*(2.*Z*E**4-0.75*E*E*Z+5.25*E)
       V(IE,IZ,3)=EEZ*(3.*Z*E**4-9.75*E*E*Z+2.25*E)
       V(IE,IZ,4)=EEZ*(2.5*Z*E**4-9.375*E*E*Z+5.625*E)
       V(IE,IZ,5)=EEZ*(2.*Z*E**3+4.5*E*Z)-12.*(C1-EEZ)
       V(IE,IZ,6)=EEZ*(3.*Z*E**3+6.75*E*Z)-24.*(C1-EEZ)
       WRITEOUTPUTTAPE6,1782,E,(V(IE,IZ,IS),IS=1,6)
 1782  FORMAT(F12.3,6E17.8)
 1704  CONTINUE
 1703  CONTINUE
C      INTEGRATION OVER G(ALPHA, ETA), ETA (20,2000)
C      G(XI) FOR ALPHA EQUALS 1.0
       GL(1,1)=.39822131E-5
       GL(1,2)=.16191809E-5
       GL(1,3)=.77825388E-6
       GL(1,4)=.41943768E-6
       GL(1,5)=.24570713E-6
       GL(1,6)=.15336414E-6
       GL(1,7)=.10062709E-6
       GL(1,8)=.68741561E-7
       GL(1,9)=.48547497E-7
       GL(1,10)=.35255710E-7
       GL(1,11)=.26218130E-7
       GL(1,12)=.19900146E-7
       GL(1,13)=.15375868E-7
       GL(1,14)=.12067421E-7
       GL(1,15)=.96028132E-8
       GL(1,16)=.77364523E-8
       GL(1,17)=.63023160E-8
       GL(1,18)=.39379518E-9
       GL(1,19)=.77591538E-10
       GL(1,20)=.24469028E-10
       GL(1,21)=.99831569E-11
       GL(1,22)=.47928589E-11
       GL(1,23)=.25742009E-11
       GL(1,24)=.15007791E-11
       GL(1,25)=.93149386E-12
```

```
      GL(1,26)=.60740136E-12
      GL(1,27)=.41219684E-12
      GL(1,28)=.28909563E-12
      GL(1,29)=.20844512E-12
      GL(1,30)=.15387694E-12
      GL(1,31)=.11592451E-12
      GL(1,32)=.88891862E-13
      GL(1,33)=.69231106E-13
      GL(1,34)=.54667366E-13
      GL(1,35)=.43701289E-13
      GL(1,36)=.35323215E-13
C     G(XI) FOR ALPHA EQUALS 1.5
      GL(2,1)=.10336716E-5
      GL(2,2)=.41716614E-6
      GL(2,3)=.19976667E-6
      GL(2,4)=.10745283E-6
      GL(2,5)=.62878519E-7
      GL(2,6)=.39223833E-7
      GL(2,7)=.25727630E-7
      GL(2,8)=.17572396E-7
      GL(2,9)=.12409252E-7
      GL(2,10)=.90115547E-8
      GL(2,11)=.67015941E-8
      GL(2,12)=.50868314E-8
      GL(2,13)=.39305232E-8
      GL(2,14)=.30849447E-8
      GL(2,15)=.24550185E-8
      GL(2,16)=.19779795E-8
      GL(2,17)=.16113962E-8
      GL(2,18)=.10074884E-9
      GL(2,19)=.19854645E-10
      GL(2,20)=.62617079E-11
      GL(2,21)=.25547955E-11
      GL(2,22)=.12265598E-11
      GL(2,23)=.65877883E-12
      GL(2,24)=.38407473E-12
      GL(2,25)=.23838560E-12
      GL(2,26)=.15544492E-12
      GL(2,27)=.10548872E-12
      GL(2,28)=.73984955E-13
      GL(2,29)=.53345039E-13
      GL(2,30)=.39380048E-13
      GL(2,31)=.29667326E-13
      GL(2,32)=.22749168E-13
      GL(2,33)=.17717610E-13
      GL(2,34)=.13990474E-13
      GL(2,35)=.11184047E-13
      GL(2,36)=.90399344E-14
C     MULTIPLY BY WEIGHTING FACTORS
      DO1710J=1,2
C     INTERVAL (20, 100)
      WGL(J,1)=GL(J,1)/0.6
      DO1711IE=2,16,2
 1711 WGL(J,IE)=GL(J,IE)/0.15
      DO1712IE=3,15,2
 1712 WGL(J,IE)=GL(J,IE)/0.3
      WGL(J,17)=GL(J,17)/0.6
C     INTERVAL (100, 900)
      WGL(J,18)=GL(J,17)*100./3.
      DO1713IE=18,24,2
      IEE=IE+1
 1713 WGL(J,IEE)=GL(J,IE)*400./3.
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      DO1714IE=19,23,2
      IEE=IE+1
 1714 WGL(J,IEE)=GL(J,IE)*200./3.
      WGL(J,26)=GL(J,25)*100./3.
C     INTERVAL (900, 1900)
      WGL(J,27)=GL(J,25)*100./3.
      DO1715IE=26,34,2
      IEE=IE+2
 1715 WGL(J,IEE)=GL(J,IE)*400./3.
      DO1716IE=27,33,2
      IEE=IE+2
 1716 WGL(J,IEE)=GL(J,IE)*200./3.
      WGL(J,37)=GL(J,35)*100./3.
 1710 CONTINUE
      DO1717IZ=1,NZ
      WRITEOUTPUTTAPE6,1783,ZE(IZ)
 1783 FORMAT(1H1F12.3)
      DO1718J=1,3
      DO1718IS=1,6
 1718 STRS(J,IS)=0.0
      DO1719IS=1,5,2
      ISS=IS+1
      DO1720IE=1,17
      STRS(1,IS)=STRS(1,IS)+WGL(1,IE)*V(IE,IZ,IS)
 1720 STRS(1,ISS)=STRS(1,ISS)+WGL(2,IE)*V(IE,IZ,ISS)
      DO1721IE=17,25
      IEE=IE+1
      STRS(2,IS)=STRS(2,IS)+WGL(1,IEE)*V(IE,IZ,IS)
 1721 STRS(2,ISS)=STRS(2,ISS)+WGL(2,IEE)*V(IE,IZ,ISS)
      DO1722IE=25,35
      IEE=IE+2
      STRS(3,IS)=STRS(3,IS)+WGL(1,IEE)*V(IE,IZ,IS)
 1722 STRS(3,ISS)=STRS(3,ISS)+WGL(2,IEE)*V(IE,IZ,ISS)
      STRS(4,IS)=STRS(1,IS)+STRS(2,IS)+STRS(3,IS)
 1719 STRS(4,ISS)=STRS(1,ISS)+STRS(2,ISS)+STRS(3,ISS)
      DO1723J=1,4
 1723 WRITEOUTPUTTAPE6,1784,J,(STRS(J,IS),IS=1,6)
C     1 REFERS TO INTEGRAL OVER THE INTERVAL (20, 100)
C     2 REFERS TO INTERVAL (100, 900)
C     3 REFERS TO INTERVAL (900,1900)
C     4 REFERS TO INTERVAL (20, 1900)
 1784 FORMAT(I4,6E17.8)
 1717 CONTINUE
      CALLSYSTEM
      END  (   1 ,   1 ,   0 ,   1 ,   0   )
```

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